The Mathematics Teacher

MARCH 1961

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XUM

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XUM

*The minimum fraction of the popular vote that can elect the President of the United States**

G. PÓLYA, *Stanford University, Stanford, California.*

"When we treat an applied problem, our first task is to give it a mathematical formulation."

I THINK THAT in the classroom an alert teacher can instructively discuss this timely question: "How small a fraction of the popular vote can elect a president?" In a final section, after formulating and solving the problem, I shall explain which feature of the question is, in my opinion, especially instructive.

FORMULATION

Our problem could be stated and solved without any or with less algebraic notation, but we would then need more time and effort to discuss it satisfactorily.

Let us use the following notation:

T is the total number of votes cast in the fifty states,

W is the number of votes received by the winning presidential candidate.

Our problem is to find the least possible value of W/T, that is, the minimum fraction of the popular vote that can elect a president. We must, however, add some conditions to render our problem mathematical and determinate.

Let r denote the number of representatives that a certain state sends to the House of Representatives, and m denote the number of those states that have exactly r representatives. Table 1 exhibits the actual values of r and the corresponding values of m. (To understand clearly the meaning of Table 1, pick out the row that refers to your state and name all the states to which this row refers; then interpret similarly a few more rows, for instance, the first row and the last.) The sum of all the values m is 50, the number of states, and the sum of all the values mr is 437, the number of congressmen. (Why?) These numbers are displayed at the bottom of Table 1. (Add the numbers in the corresponding column.)

The number of votes cast in a state is

TABLE 1

COMPOSITION OF THE HOUSE OF REPRESENTATIVES

r	m	mr
1	6	6
2	9	18
3	1	3
4	3	12
6	7	42
7	2	14
8	4	32
9	3	27
10	3	30
11	2	22
12	1	12
14	2	28
18	1	18
22	1	22
23	1	23
25	1	25
30	2	60
43	1	43
Totals	50	437

* From the "Seminar in Problem Solving" conducted by the author in the National Science Foundation Academic Year Institute at Stanford University.

approximately proportional to its population, which is approximately proportional to the number r of its representatives. To arrive at a problem which is both simple and precise, we *assume* that the *number of votes cast in a state is* (not only approximately, but) *exactly proportional to the number* r *of its representatives.* Hence, there is a (large, fixed) number N such that in any state that has r representatives the number of votes cast in the presidential election is rN. Hence, the total popular vote

$$T=437N. \tag{1}$$

We introduce a *second assumption:* there are *only two presidential candidates,* and any vote counted in the total number T of votes is cast for one of them. As the winner receives W votes, his adversary receives $T-W$ votes.

For each state, the number of electors sent to the Electoral College equals the sum of the state's representatives and senators. A state that has r representatives, has $r+2$ votes in the Electoral College (where the total number of votes is, therefore, $437+100=537$). We make a *third assumption: any state gives all its votes to the candidate* who carries it, that is, *who obtains a majority of votes in the state.*

We have made just three assumptions. They seem to me simple and "realistic," that is, as close to actual circumstances as their simplicity allows. At any rate, these three assumptions yield a numerically determined answer to our question.

Solution

The winner must obtain more than one half of the 537 votes of the Electoral College, that is, at least 269 votes. Let us say the winner carries s states, with $r_1, r_2, \cdots r_s$ representatives respectively; then, necessarily,

$$(r_1+2)+(r_2+2)+\cdots+(r_s+2)\geqq 269$$

or

$$r_1+r_2+\cdots+r_s\geqq 269-2s. \tag{2}$$

To carry a state with $r+2$ electoral votes, the winner must obtain more than half the votes cast in that state, that is, at least $(r\ N/2)+1$ votes. (N is even, let us assume.) To carry the s states mentioned, the total number W of votes received by the winner must satisfy the inequality

$$W\geqq(\tfrac{1}{2}r_1N+1)+(\tfrac{1}{2}r_2N+1)+\cdots+(\tfrac{1}{2}r_sN+1)$$

or

$$W\geqq\frac{N}{2}(r_1+r_2+\cdots+r_s)+s. \tag{3}$$

From (1) and (3) we obtain information about the fraction with which we are concerned

$$\frac{W}{T}\geqq\frac{r_1+r_2+\cdots+r_s}{874}+\frac{s}{437N}$$

and hence, using (2),

$$\frac{W}{T}\geqq\frac{269-2s}{874}+\frac{s}{437N}. \tag{4}$$

Can the case of equality be attained in (4)? This question is crucial, and here is the answer: equality in (4) can be attained if, and only if, equality is attained in both relations (2) and (3), from which we have derived (4). Now, equality in (2) is attained if the states carried by the winner have jointly just 269 electoral votes. And equality in (3) is attained if the winner has the barest majority in the s states he has carried, and no votes at all in the remaining $50-s$ states. This extreme situation is compatible with our assumptions, and, although it can scarcely arise in an actual election, we may examine with some interest how much it differs from the actual results of elections.

Hence, the minimum value of W/T will be attained if equality is valid in (4) *and* the maximum value of s is attained. Therefore, we have to collect the greatest possible number s of states which have jointly precisely 269 electoral votes. Obviously, in collecting them we should

TABLE 2*

OBTAINING THE LONGEST LIST (SET) OF STATES HAVING PRECISELY 269 ELECTORAL VOTES

$r+2$	m	Σm	$m(r+2)$	$\Sigma m(r+2)$
3	6	6	18	18
4	9	15	36	54
5	1	16	5	59
6	3	19	18	77
8	7	26	56	133
9	2	28	18	151
10	4	32	40	191
11	3	35	33	224
12	3	38	36	260
13	2			

* The top entry in the column with the heading Σm is the top number in the preceding column; any other entry is the sum of two numbers, the one immediately to the left and the other immediately above. The columns with headings $\Sigma m(r+2)$ and $m(r+2)$ are related to each other in the same way.

start with the least populous states, each having just one representative, and then pass successively to the higher values 2, 3, 4, $\cdots$ of r. The necessary simple computations are displayed in Table 2, which should be self-explanatory (but study it before you present it to your class). We find that there are 38 states each with no more than twelve electoral votes apiece and that these 38 states have jointly 260 electoral votes. Remove from these 38 states one which has 4 electoral votes and add one which has 13 electoral votes; the set of 38 states so obtained commands precisely 269 electoral votes. Any set of 39 states, however, has at least 273 electoral votes, and so the desired maximum value of s is 38. Hence, the minimum value of the fraction of the popular vote that can elect a president, see (4), is

$$\frac{269-76}{874}+\frac{38}{437N}=\frac{193}{874}+\frac{38}{437N}=0.220824.$$

In the last numerical value I neglected the term containing N. In fact, the total popular vote in the last election was more than 68 million. If we equate this to 437 N which is, according to our assumption, the total popular vote, see (1), we observe that the neglected term would contribute less than one unit to the sixth decimal.

It will surprise some of us that within the framework of the present laws and with population figures not very different from those of the last census, in some freak political constellation, a minority of just a little over 22 per cent could elect the president.

APPLIED PROBLEMS

Few high school students will become mathematicians, but many more will become users of mathematics. Mathematics is used to solve *applied problems*, that is, practical or scientific problems which are not purely mathematical. Does the high school pay sufficient attention to the needs of prospective users of mathematics?

The problem of formulating a problem. When we treat an applied problem, our first task is to give it a mathematical formulation—express it in mathematical language, reduce it to mathematical concepts. This first task may easily be the most important, the heaviest with consequences, and the most delicate. In fact, in reducing an applied problem to mathematical form, we are in great danger of committing one of two opposite mistakes; we may do too much or too little, we may sin by transgression or by omission. Reality is infinitely complex, and so is even that little part of it that our problem intends to master. Hence, in passing to a mathematical formulation, we are obliged to neglect, to simplify, to idealize. We may err by neglecting too much, and then our mathematical problem becomes unrealistic, out of touch with that little part of nature that we intended to master. Or we may err in neglecting too little, in trying to keep too many details, and then our problem may become unmanageable or unbalanced, too formidable or too expensive to solve. In fact, to give a nontrivial applied problem an appropriate mathematical formulation which is neither unrealistic nor inaccessible may be a major

achievement which needs everything: experience, knowledge, talent, all the art of the "applied mathematician"—and good luck, too.

How many decimals? It happens only exceptionally that we can solve an applied problem "exactly," "explicitly." Most of the time, applied problems are solved by approximation. And then the question arises (and it is an important question): How far should the approximation go? How many decimals should we compute?

A reasonable answer depends on the mathematical formulation of the applied problem: a more realistic formulation deserves more work, deserves an approximation pushed further, than does a less realistic formulation. Aiming at a preassigned number of decimals has sense only if we can reasonably estimate the influence of neglected details.

And how about the high school? Each high school algebra course offers a good opportunity to do something essential for prospective users of mathematics (an opportunity insufficiently exploited, even insufficiently recognized, I think) when it presents "word problems." In fact, a high school student who "sets up equations" to solve a "word problem" reduces some factual situation to mathematical concepts and he thus has a first taste of the most essential activity of the serious user of mathematics. Yet the traditional word problems, even if put to a better use, give little opportunity to discuss such important questions as, "Is this formulation sufficiently realistic?" or "How many decimals are meaningful?" Problems in science, especially in elementary mechanics, give a good opportunity to discuss such questions, but high school students are not often sufficiently prepared.

I consider it as a merit of the problem presented in the foregoing sections that the students of an average high school class can *actively participate in the formulation* and, in doing so, can discuss with understanding (even with heat) the question, "Is this assumption realistic?" At the end of the solution, another question, "Is this decimal meaningful?" can be usefully discussed.

To teachers who intend to present the problem in their classes, I want to say this: Please, before presenting the problem, consider carefully the mode and level of presentation—with as much algebra as here, with less algebra, or without algebra; with formal conclusions as here, or more "popularly," more "intuitively."

I cannot suppress a remark on traditional textbook word problems: they have so often exasperated so many of us by supposing exact proportionality where it is out of place and by employing similar tricks. They are often so glaringly unrealistic that they are just bound to shock the student with a good practical mind and give him a feeling of contempt for mathematics. I wish to submit the following for your consideration: "If two authors need three years to write a book, in how much time will the book be written by 365 authors?"

See you in Chicago
April 5–8
at the
Thirty-ninth Annual Meeting
to be held at the Conrad Hilton Hotel.

A chain of circles

RODNEY T. HOOD, *Franklin College, Franklin, Indiana.*

"If we are asked a question in Russian, we may need an interpreter to turn it into English. If we can then answer the question in English, we can give our reply to the interpreter, who can turn it back into Russian."

IN THE FEBRUARY, 1960, issue of THE MATHEMATICS TEACHER, John Satterly presented an interesting problem involving a series of touching circles. Briefly, the problem is that of inscribing a chain of tangent circles within the curve consisting of three mutually tangent semicircular arcs, with radii r, r, and $2r$. The solution is carried out for several steps by the rather lengthy methods of analytic geometry, and the reader is informed that further circles of the chain can be obtained in similar fashion. We are also told that the problem can be solved by inversion, but the details of the process are not given.

Since this problem of the circles can be handled so neatly by inversion and at the same time generalized without difficulty, and since, moreover, it is a problem having not only historical interest but pedagogical value as well in the use of a geometrical transformation, it seems appropriate to consider it anew.

We begin with a base line l and choose on it any two points O and P. On a given side of l we construct a semicircle S_1 on OP as diameter. On this diameter we choose any point Q and construct semicircles S_2 and S_3 on OQ and QP, respectively. The three semicircles are all erected on the same side of l. If the radius of S_i is r_i $(i=1, 2, 3)$, then we have

$$r_1=r_2+r_3.$$

The resulting figure (which reduces to Professor Satterly's, when $r_2=r_3$) is an arbelos ("shoemaker's knife," so-called from its shape), a famous curve in the history of geometry (Fig. 1). It appears in the third century B.C. in the writings of Archimedes and was subsequently studied by Pappus and later geometers. It has a number of interesting properties.[1]

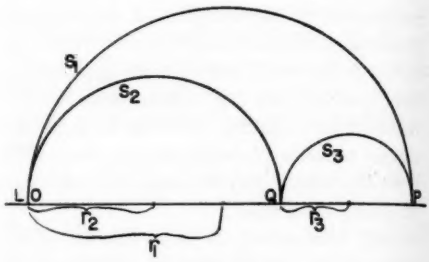


Figure 1

Ancients and moderns alike have been interested in the problem of constructing in the arbelos a chain of tangent circles $C_1, C_2, C_3, \cdots$, such that (for every i) C_i is tangent to S_1 and S_2 and also to C_{i-1} (or to S_3, if $i=1$). We shall develop a general solution to this problem which will apply to an arbelos of any size and shape. Our method makes use of the transformation known as *inversion*, developed in the early part of the nineteenth century.

The use of a geometrical transformation in solving a construction problem is somewhat like the use of an interpreter in ordinary discourse. If, for example, we are asked a question in Russian, we may need an interpreter to turn it into English. If

[1] See, for example, Howard Eves, *An Introduction to the History of Mathematics* (New York: Rinehart and Co., 1953), p. 166.

we can then answer the question in English, we can give our reply to the interpreter, who can turn it back into Russian. We thus have the answer to the original question in the appropriate language.

Here, inversion plays the role of the interpreter. Using it, we seek to transform the arbelos into a related figure, and the problem into a related problem which can be solved. Then by a reverse process, which in this case is simply a second use of inversion, we return to the original arbelos and have forthwith the desired solution. To make the process clear, we first define inversion and note some of its properties.

This transformation makes use of a basic circle C (the circle of inversion) and its center O (the center of inversion, or origin). Then, under inversion with respect to C, any point X (except O itself) is transformed into an inverse point X', such that O, X, and X' are collinear, with X and X' on the same side of O, and so situated that

$$OX \cdot OX' = r^2, \tag{1}$$

where r is the radius of C. Similarly, any curve K will be transformed into an inverse curve K'—the set of all the inverses of the points of K. From (1) it is clear that if X' is inverse to X, X is also inverse to X', and hence that the inverse of the inverse of any geometrical figure is the figure itself.

It is a remarkable fact that if K is any circle or straight line (which we might regard as a circle of infinite radius), then K' is also a circle or straight line. Specifically, we have these properties:

a) If K is a straight line through O, K' is the same straight line.

b) If K is a circle through O, K' is a straight line not through O, perpendicular to the diameter of K which goes through O.

c) If K is a circle not through O, K' is a circle not through O. Usually K' is distinct from K. However, the inverse of C is C, because each point of C is its own inverse. The same is true for certain other circles.

d) If K_1 and K_2 are two curves with a common point X, K_1' and K_2' will have the common point X', and the angles at X and X' between corresponding arcs will be equal. We express this property by saying that the transformation is *conformal*. Among other things, this implies that the inverses of two tangent curves will be tangent curves, and the inverses of orthogonal curves (curves intersecting at right angles) will be orthogonal.

With this background, we can proceed to invert the arbelos. We choose O as the center of inversion. For C, any radius may be used; however, it is convenient to have P and Q inverse points. That is, we choose r so that $OP \cdot OQ = r^2$. It is not necessary actually to construct C.

With this choice, the base line of the arbelos is inverted into itself, by property (a). The semicircles S_1 and S_2, being the upper halves of circles through O, become the upper halves of lines perpendicular to L, by (b). Since S_1 and S_2 pass through P and Q, respectively, their inverses will pass through $P'(=Q)$ and $Q'(=P)$. Since S_3 passes through P and Q at right angles to L, its inverse will be the upper half of a circle through Q and P at right angles to L, by (c) and (d). Since there is only one such semicircle, it follows that S_3 is its own inverse. Thus the inverted figure consists of a semicircle and two rays tangent to it at the ends of its diameter (Fig. 2).

The circles C_i are to be tangent to each other and to S_1 and S_2. Obviously, none of them will pass through O. Therefore, by (c), their inverses C_i' will also be circles. By (d), these inverse circles will be tangent to each other and to S_1' and S_2'. That is, they will be a chain of circles tangent to the vertical lines through Q and P. Each will have a radius r_3 and will be tangent to the lines at points equally spaced (a distance $2r_3$ apart). They will also be tangent to each other at points equally spaced along a line m' which is parallel to S_1' and S_2' and lying midway between them.

Let the ith circle C_i' be tangent to S_1'

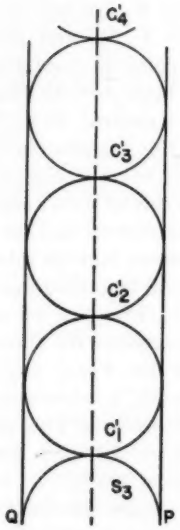


Figure 2

at the point P_i' and to S_2' at the point Q_i'. Clearly, their inverses P_i and Q_i will be on the inverses of S_1' and S_2', that is, on S_1 and S_2, respectively. In addition, P_i lies on OP_i' and Q_i lies on OQ_i'. Thus P_i and Q_i can easily be determined. The circle C_i passes through these points; moreover, they are the points at which C_i touches the semicircles. Now when two circles are tangent, they touch at a point on their common line of centers. Therefore, the center of C_i will lie on each of lines AP_i and BQ_i, where A and B are the centers of the semicircles (Fig. 3). We thus determine O_i, the center of C_i. We can accordingly construct C_i itself without difficulty, using either O_iP_i or O_iQ_i as radius. We note that it is not necessary actually to construct the circle C_i'. Nor do we need to obtain $C_1, C_2, \cdots, C_{i-1}$ first, to construct C_i. In this way we can construct any circle of the chain.

The above solution is theoretically perfect, of course. In practice, however, the two lines AP_i and BQ_i often make such a small angle with each other that their point of intersection cannot be found with sufficient precision to yield a good result.

Therefore, as a check and supplement to these two lines, we may make use of the line of centers of two consecutive members of our chain. We recall that C_i' is tangent to C_{i-1}' (or to S_3, if $i=1$) at a point which we shall call T_i'. All these points T_i' lie on a line m'. Therefore, their inverses T_i lie on the inverse of m', which by (b) is a circle (actually, just the upper half circle) through O, which we shall call m. To determine m, we note that it contains the point T_1, the inverse of T_1'. But T_1' is also on S_3. Hence, T_1 is likewise on S_3, since as we have seen S_3 is its own inverse. Therefore, we construct the line OT_1' and locate T_1 at the other point of intersection of this line with S_3. We now construct the perpendicular bisector of OT_1; the point where this bisector meets l will be obviously the center of m.

Once semicircle m is obtained, we can find the points T_i on it, because they will be collinear with O and T_i'. Then, once a circle C_{i-1} has been constructed, the center O_i of the following circle will be collinear with the known center O_{i-1} and the known point of tangency T_i.

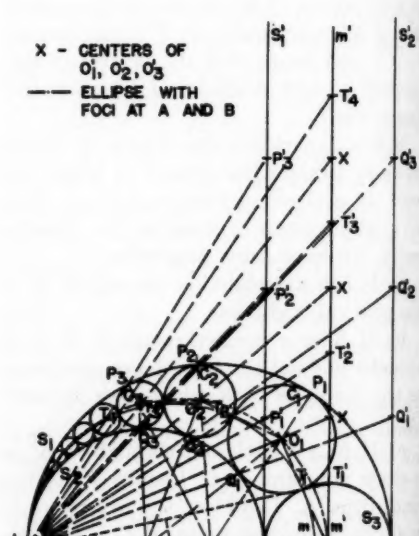


Figure 3

We may pause to recall a "nonproperty" of inversion. In general, the centers of two inverse circles are not themselves inverse points. Thus, although the centers of the circles C_i' lie on the line m', the centers of the circles C_i do not lie on m. As Professor Satterly remarks in his article, they seem to lie almost but not quite on a circle. What more can we say about them?

Since O_i is the center of C_i, we have $O_iP_i = O_iQ_i$, or equivalently

$$AP_i - AO_i = BO_i - BQ_i.$$

It follows that

$$(2) \quad AO_i + BO_i = AP_i + BQ_i = r_1 + r_2 = \text{a constant for all } i.$$

Therefore, the centers of our chain lie on the upper half of an ellipse with foci at A and B and with major axis equal to the sum of the radii of S_1 and S_2. Indeed, (2) is just the condition that a point O_i should lie on this curve.

Granting the "nonproperty" of inversion mentioned above, we may nevertheless observe that the centers of C_i and C_i' are collinear with the origin. This is clear from the symmetry of the circles involved with respect to their common line of centers. We may make use of this fact as an additional control in seeking the practical determination of the points O_i. We can also derive several simple formulas in connection with the points P_i, Q_i, T_i, and O_i, as a consequence of the collinearity of pairs of inverse points with the origin and the fact that in the inverted arbelos the points P_i', Q_i', T_i', and the centers of C_i' are regularly spaced on the vertical lines S_1', S_2', and m'. If p_i, q_i, t_i, and o_i are the slopes of the lines OP_i, OQ_i, OT_i, and OO_i, respectively (relative to the base line L), it follows immediately that these relations hold:

$$p_i = ip_1,$$
$$q_i = iq_1,$$
$$t_i = (2i-1)t_1,$$
$$o_i = io_1 = 2it_1, \quad (i = 1, 2, 3, \cdots)$$

In conclusion, it may be observed that our problem of the circles is a special case of the problem of constructing circles tangent to any three given circles, which may not themselves be tangent. This general problem can also be solved by inversion.[2] It, too, is a problem of great antiquity, going back to "the great geometer" Apollonius, a younger contemporary of Archimedes, who wrote the book *On Tangencies*, now lost. Finally, inversion can also be used for filling with additional circles the interstices between our circles O_i and S_1 and S_2 and S_3, and so on ad infinitum (et absurdum!). Only the endurance of the student can set limits to the problem!

[2] For a brief treatment, see Richard Courant and Herbert Robbins, *What Is Mathematics? An Elementary Approach to Ideas and Methods* (London: Oxford University Press, 1941), p. 161.

Have you read?

DRENICK, R. F., "Random Processes in Control and Communications," *Science*, September 30, 1960, pp. 865–870.

What changes have taken place in the applications of mathematics in the past fifteen years? Read this article for one interesting application to the "signal space." What does the engineer do first? He goes to the mathematician and explains his problem. What does the mathematician do? He says he suspects the random process, he explains the idea of a point in infinite-dimensional space, and he covers the space with a probability measure. Then he springs the "set" on the engineer with its unions and intersections and soon has led him into families of sets called "Borel fields."

Then the engineer and the mathematician engage in "Mathematical Archeology," taking layer after layer off the Borel field but always some remains. The problem of converting one signal to another is faced by a linear transformation. Here are "Wiener's Prediction Theory" and "Shannon's Information Theory"—and what has happened to the engineer? He is back in the lab trying to reduce to practice the theory of the mathematician. A fascinating article.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

An analysis of the quadratic

JOSEPH F. HOHLFELD, *Iowa State Teachers College, Cedar Falls, Iowa.*

Some interesting properties of quadratic equations are discussed here.

IN TEACHING QUADRATIC EQUATIONS we frequently restrict ourselves to a study of their roots by factoring, completing the square, and the quadratic formula. Sometimes we go on to analysis of these roots through a study of the discriminant. While the importance of the study of roots is not to be minimized, considerable benefit may be derived from a graphic analysis of the quadratic function. This paper is mainly concerned with some statements which can easily be made about the graph of a condition of the form $y=ax^2+bx+c$, the reasoning behind these statements, and a few generalizations about how the choice of replacements for a, b, and c affect the parabola. Since the roots of a quadratic equation are the x co-ordinates of the points where the corresponding parabola crosses the x axis, this concept is strengthened, although roots are of secondary concern in this paper.

It should be noted that the first derivative is used in this development because it seems to be the simplest and most direct approach. In the high school classroom, it would probably be better to approach this intuitively. On the other hand, if the teacher feels rigor is necessary, these conclusions can be reached without the calculus. For example, part of what is done here can be accomplished by completing the square.

From the calculus we know that the first derivative of a function may be interpreted as the *slope* of the tangent to the graph of the function at any point. Consider the sentence $y=ax^2+bx+c$ and its first derived sentence

$$\frac{dy}{dx}=2ax+b.$$

For the replacement

$$x=0, \qquad \frac{dy}{dx}=2a(0)+b, \qquad \text{so } \frac{dy}{dx}=b.$$

Thus the tangent to the graph of $y=ax^2+bx+c$ at the point $(0, c)$ has a slope of b. Therefore, the equation of the tangent to the curve at $(0, c)$ is $y=bx+c$. We then have our first conclusion, and it is deserving of emphasis.

Conclusion 1: For each replacement of a, b, and c, $y=bx+c$ is tangent to $y=ax^2+bx+c$ at the point $(0, c)$.

It is interesting to note that when $a=0$, $y=ax^2+bx+c$ degenerates into $y=bx+c$.

Again from the calculus we know that the graph of $y=ax^2+bx+c$, $(a\neq 0)$ has exactly one maximum or minimum value of y, and that this will occur at a value of x for which the first derivative is equal to zero.

$$2ax+b=0;$$

therefore

$$x=\frac{-b}{2a}.$$

If

$$y=ax^2+bx+c \text{ and } x=\frac{-b}{2a},$$

then

$$y=a\left(\frac{-b}{2a}\right)^2+b\left(\frac{-b}{2a}\right)+c;$$

hence

$$y=\frac{-b^2}{4a}+c.$$

Thus we come to another conclusion.

Conclusion 2: The maximum or minimum point of the parabola that corresponds to the quadratic condition $y=ax^2+bx+c$, $(a\neq 0)$ is

$$\left(\frac{-b}{2a}, \frac{-b^2}{4a}+c\right).$$

It can also be shown that the line whose equation is

$$x=\frac{-b}{2a},$$

$(a=0)$ is its axis of symmetry.

The proof of whether a given quadratic function has a maximum or minimum y-value is an exercise in algebra and can be given as follows.

Consider the set of all (x, y) such that

$$y=ax^2+bx+c$$

and

$$x=-\frac{b}{2a}+f.$$

From this we can solve for y.

$$y=a\left(\frac{-b}{2a}+f\right)^2+b\left(\frac{-b}{2a}+f\right)+c$$

$$y=a\left(\frac{b^2-4abf+4a^2f^2}{4a^2}\right)-\frac{b^2-2abf}{2a}+c$$

$$y=\frac{4a^2f^2-b^2}{4a}+c$$

$$y=af^2-\frac{b^2}{4a}+c.$$

Now if $f=0$ then

$$af^2-\frac{b^2}{4a}+c=\frac{-b^2}{4a}+c.$$

When $f\neq 0$, $f^2>0$ and if $a>0$ then

$$af^2-\frac{b^2}{4a}+c>\frac{-b^2}{4a}+c;$$

if $a<0$ then

$$af^2-\frac{b^2}{4a}+c<\frac{-b^2}{4a}+c;$$

and if $a=0$, both terms are undefined.

Hence, the third conclusion follows.

Conclusion 3: When a is positive, a *minimum* value of y occurs at the point

$$\left(\frac{-b}{2a}, \frac{-b^2}{4a}+c\right).$$

When a is negative, a *maximum* value of y occurs at the point

$$\left(\frac{-b}{2a}, \frac{-b^2}{4a}+c\right).$$

When a is zero, a maximum or minimum y-value is undefined.

The sentence

$$y=\frac{-b^2}{4a}+c$$

can be written

$$y=\frac{b}{2}\left(\frac{-b}{2a}\right)+c.$$

Since at a maximum or minimum

$$x=\frac{-b}{2a},$$

special significance can be attached to the line

$$y=\frac{b}{2}x+c,$$

considering a as a parameter and holding b and c constant.

Conclusion 4: For each replacement of a, the maximum or minimum point of the parabola for $y=ax^2+bx+c$ lies on the line

$$y=\frac{b}{2}x+c.$$

Let us consider the implications of the foregoing conclusions to a specific case. Consider $y=\frac{1}{4}x^2+2x+3$.

Conclusion 1: The parabola $y=\frac{1}{4}x^2+2x+3$ is tangent to

$$y=2x+3 \quad \text{at} \quad (0, 3).$$

Conclusion 2: Since $\frac{1}{4}$ is positive, this parabola has a minimum.

Conclusion 3: This minimum occurs at

$$\left(\frac{-2}{2\cdot(\frac{1}{4})}, -\frac{(2)^2}{4\cdot(\frac{1}{4})}+3\right)=(-4,-1)$$

and the axis of symmetry of the parabola is the line $x=-4$.

Conclusion 4: The point $(-4, -1)$ lies on the line

$$y=\frac{2}{2}x+3$$

or simply $y=x+3$.

These conclusions are graphically illustrated in Figure 1, not only for $y=\frac{1}{4}x^2+2x+3$, but also for the following other members of the $y=ax^2+2x+3$ family of parabolas.

$$y=x^2+2x+3$$
$$y=\tfrac{1}{4}x^2+2x+3$$
$$y=-\tfrac{1}{4}x^2+2x+3$$
$$y=-1x^2+2x+3$$

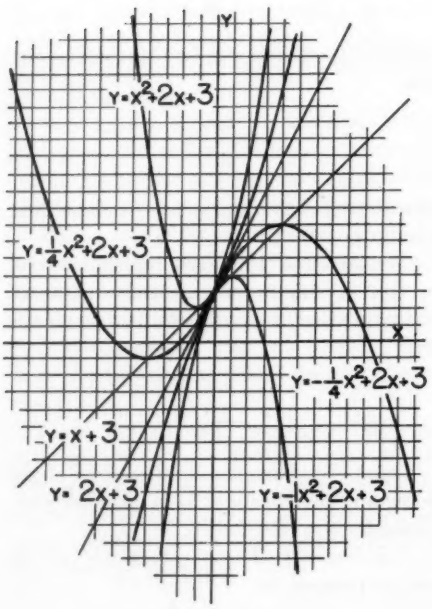


Figure 1

We can now recognize a pattern in the graph. When $|a|$ is very large, the parabola is relatively narrow and the maximum or minimum is very close to the y-axis on the line $y=x+3$. When a is positive, the parabola has a minimum, and when a is negative, it has a maximum. As $|a|$ becomes smaller, the maximum or minimum point moves farther and farther away from (0, 3) along the line $y=x+3$. That is, when a is positive, for a smaller value of a the minimum is farther to the left along the line $y=x+3$, and when a is negative and getting larger ($|a|$ getting smaller) the maximum is farther to the right along the line $y=x+3$. As a approaches 0 from either direction these points move farther and farther out until, when $a=0$, the parabola for $y=ax^2+2x+3$ degenerates into the straight line $y=2x+3$.

When b is positive, the minimums lie to the left of the y-axis and the maximums to the right. As b becomes smaller,

$$y=\frac{b}{2}x+c$$

rotates clockwise. As $|b|$ becomes smaller, the maximum and minimum points move closer to (0, 3) until, when $b=0$, they coincide at (0, 3). As b becomes negative, the maximums lie to the left of the y-axis and the minimums to the right.

The varying of c is simplest of all, merely changing the y-intercept and moving the whole set of curves up and down.

For those who approach the parabola from the focal point and directrix we can carry this further. Let us first consider the case of the parabola that corresponds to the quadratic condition $y=ax^2$. (See Fig. 2.) In this case, the y-axis is its axis of symmetry and the focal point, F, is p units above the x-axis while the directrix lies p units below. If Q is a point on the parabola, its co-ordinates are (x, ax^2). We see that M is a point on the directrix and QM is perpendicular to the directrix at M.

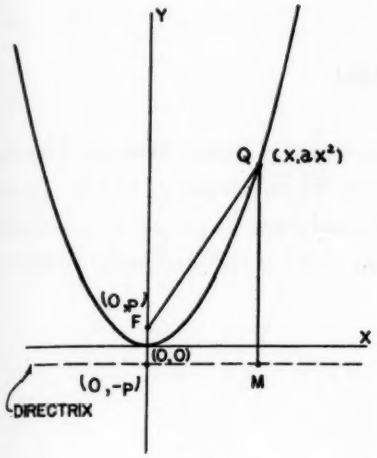


Figure 2

Then by definition $QM = FQ$. By the distance formula

$$FQ = \sqrt{x^2 + (ax^2 - p)^2}$$

$$QM = ax^2 + p.$$

Then

$$ax^2 + p = \sqrt{x^2 + (ax^2 - p)^2}.$$

Solving for p we find

$$p = \frac{1}{4a}.$$

Therefore we find that F for the general quadratic $y = ax^2 + bx + c$ lies at the point

$$\left(-\frac{b}{2a}, -\frac{b^2}{4a} + c + \frac{1}{4a}\right).$$

There are many simpler forms for this expression. The simplest for our purpose is

$$\left(\frac{-b}{2a}, \frac{1-b^2}{4a} + c\right).$$

Now the influence of bx and c on F is restricted to their influence on the position of the entire parabola in the plane. That is, F will always lie on the axis of symmetry, and it will always lie

$$\frac{1}{4a}$$

units above the minimum. We have the co-ordinates for the minimum

$$\left(-\frac{b}{2a}, -\frac{b^2}{4a} + c\right)$$

from our second conclusion. Note that the focal points for a given parabola will lie a vertical distance of

$$\frac{1}{4a} \text{ above the line } y = \frac{b}{2}x + c.$$

Therefore, the line

$$y = \frac{b}{2}x + c + \frac{1}{4a}$$

is the set of all the points which are focal points of all parabolas of quadratic conditions of the form $y = ax^2 + bx + c$.

The sentence for the directrix then is

$$y = \frac{-b^2}{4a} + c - \frac{1}{4a} \quad \text{or} \quad y = -\frac{b^2+1}{4a} + c.$$

Regardless of the depth to which a teacher investigates the quadratic with his students, he will guide the study with more confidence and meaning if he has the total picture in mind.

"Some communities are providing honors classes and seminars for bright pupils on Saturdays or during after-school hours."—*From "Schools in Our Democracy," Office of Education, U.S. Department of Health, Education, and Welfare.*

Graphic solution of a quadratic equation

CARL N. SHUSTER, *University of Tampa, Tampa, Florida.*

After the co-ordinates of the minimum of $x^2+bx+c=0$ *have been estimated from its graph, it is possible to estimate the roots of this quadratic equation.*

LET THE ROOTS of a quadratic equation $x^2+bx+c=0$ be r_1 and r_2. Then the equation can be written $x^2-(r_1+r_2)x+r_1r_2=0$. From this we have

$$\frac{dy}{dx}=2x-(r_1+r_2).$$

Setting

$$\frac{dy}{dx}=0,$$

then $2x-(r_1+r_2)=0$, and the x of the minimum point, x_m, is (Fig. 1)

$$x_m=\frac{r_1+r_2}{2}.$$

Also,

$$y_m=f(x_m)=\left(\frac{r_1+r_2}{2}\right)^2$$

$$-\frac{(r_1+r_2)(r_1+r_2)}{2}+r_1r_2,$$

and

$$y_m=-\left(\frac{r_1-r_2}{2}\right)^2 \text{ or } -y_m=\left(\frac{r_1-r_2}{2}\right)^2.$$

At the minimum point the co-ordinates are

$$(x_m, y_m) \text{ or } \left[\frac{r_1+r_2}{2}, -\left(\frac{r_1-r_2}{2}\right)^2\right].$$

Solving the quadratic equation $x^2-(r_1+r_2)x+r_1r_2=0$ for x we get

$$x=\frac{r_1+r_2}{2}\pm\sqrt{\left(\frac{r_1-r_2}{2}\right)^2}.$$

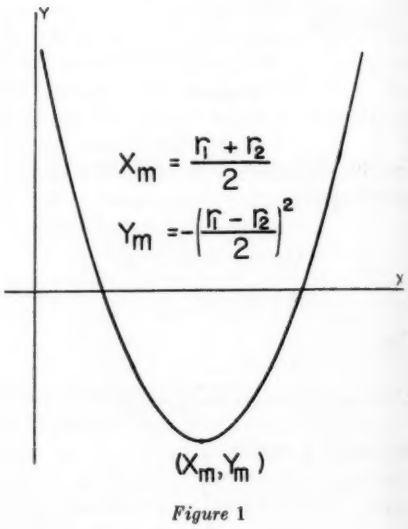


Figure 1

Thus

$$\frac{r_1+r_2}{2}+\frac{r_1-r_2}{2}=r_1.$$

and

$$\frac{r_1+r_2}{2}-\frac{r_1-r_2}{2}=r_2.$$

Therefore, if the co-ordinates of the minimum point on the graph of a quadratic equation are

$$x_m=\frac{r_1+r_2}{2} \text{ and } y_m=-\left(\frac{r_1-r_2}{2}\right)^2,$$

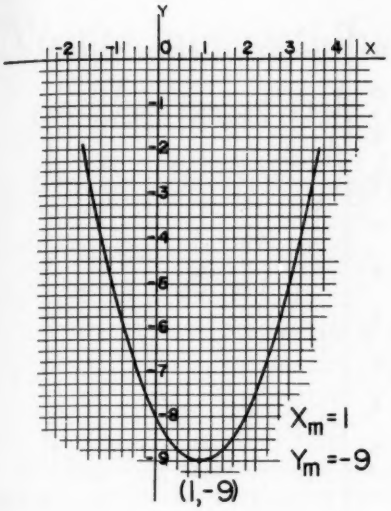


Figure 2

then

$$x = \left(\frac{r_1 + r_2}{2}\right) \pm \sqrt{\left(\frac{r_1 - r_2}{2}\right)^2},$$

or $x = x_m \pm \sqrt{-y_m}$. Therefore, if it is possible to estimate rather closely the co-ordinates of the minimum point in graphing the quadratic equation $x^2 + bx + c = 0$, then these co-ordinates may be substituted into the formula $x = x_m \pm \sqrt{-y_m}$ to find the roots, either real or imaginary. If the derivative is used to find x_m and y_m, the roots may be found to any desired degree of accuracy.

This formula will be used to solve the equations that follow.

1. $x^2 - 2x - 8 = 0$. (See graph in Figure 2.)

$x_m = 1$, $y_m = -9$, $x = x_m \pm \sqrt{-y_m}$, $x = 1 \pm 3$

$x = -2$ or 4

2. $x^2 - 6x + 9 = 0$

From the graph in Figure 3, it is noted that

$$y_m = 3, \qquad y_m = 0.$$
$$x = x_m \pm \sqrt{-y_m}$$
$$x = 3 \pm 0$$

3. $x^2 - 6x + 12 = 0$

From the graph in Figure 4 it is noted that

$$x_m = 3, \qquad y_m = 3.$$
$$x = x_m \pm \sqrt{-y_m}$$
$$x = 3 \pm \sqrt{-3}$$
$$x = 3 \pm i\sqrt{3}$$

This method has special merit in Case 3. In graphing $y = x^2 + bx + c$ it may be noted

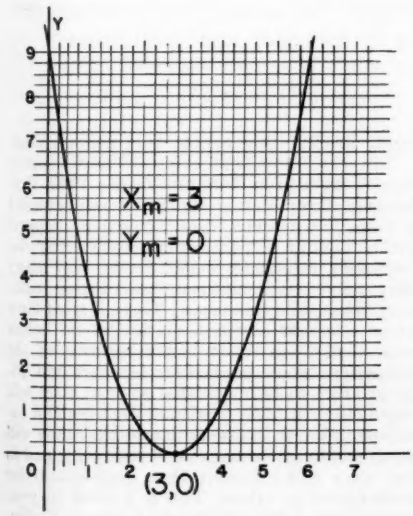


Figure 3

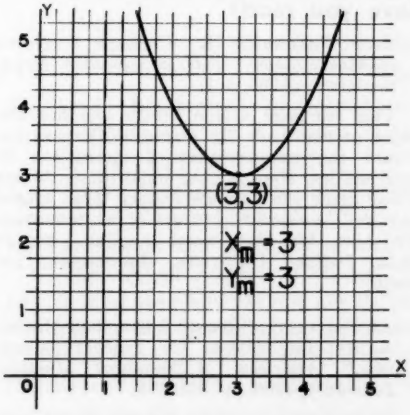


Figure 4

that the graph will not intersect the x-axis. The precedure is the same. Even though the roots are not real, they can be found from the formula.

The method given above may be used with a cubic equation. Given: $x^3-4x^2-x+22=0$. Graphing produces the real root $x=-2$. When this root is divided out, the quadratic $x^2-6x+11=0$ remains; graphing this we get Figure 5.

$$x_m=3, \qquad y_m=2.$$

$$x=x_m\pm\sqrt{-y_m}$$

$$x=3\pm\sqrt{-2}$$

$$x=3\pm i\sqrt{2}$$

To obtain the formula $x=x_m\pm\sqrt{-y_m}$ the derivative was used. It is easy to find x_m and y_m graphically so that no calculus is needed to solve quadratics by the above formula. Many high school teachers, however, include enough simple work with the derivative to enable them to use the proof given with the better students.

In the past, many texts and teachers have taught that it is impossible to find the roots of a quadratic equation graphically when the roots are imaginary. The chief purpose of this paper is to show that imaginary roots as well as positive roots may be found by using a graph.

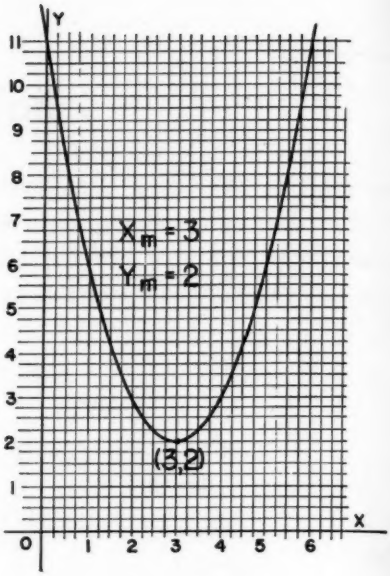


Figure 5

Have you read?

JACKSON, FREDERICK H., "Carnegie Programs in Mathematics," *Science Education News*, September, 1960, p. 1.

This one-page article brings together the major grants made by Carnegie Corporation toward the improvement of instruction in mathematics. Here is your opportunity to see in one place the possible impact these grants may well have on the future of mathematics. It's short, but well worth reading.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

JONES, PHILLIP S., "Recent Research in Mathematics: Implications for Teacher Education," *American Mathematical Monthly*, June-July, 1960, pp. 585–590.

As teachers or teachers of teachers, we must know what is going on in the mathematical world. Briefly, this article tells us just that.

Did you know that about 15 mathematics journals have been introduced since 1940, that the range of topics in mathematics has increased by leaps and bounds, that psychology departments, biology departments, engineering departments, economics departments, and many others have mathematicians as regular members? What does this mean? There is need for a better selection of mathematics to be better understood by a much larger segment of the population. There is need for teachers and administrators to be motivated so that they will continually carry on curricular changes and experimentation in mathematics, so that they will acquire more mathematics and pedagogy, and so that they will transmit their enthusiasm for mathematics to others. This is a must on your reading list.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

Ninety suggestions on the teaching of mathematics in the junior high school

EDWIN J. SWINEFORD, *University of California (Santa Barbara), Goleta, California.*

A check list of suggested activities that a junior high school mathematics teacher may use in self-evaluation.

The following items indicate specific things to do in the teaching of mathematics. These suggestions come from many sources. You may review them for your own purposes or have an observer use them as a check list for reactions to your teaching. Feel free to agree or disagree with these suggestions, or to extend and modify them for your use.

On preparing to teach mathematics

1. Secure a copy of the state curriculum guide and study the section on mathematics.
2. Study several courses of study on mathematics from other school systems. Examine Resource Units and Teaching Units on mathematics.
3. Examine several textbooks on the teaching of mathematics and several used by students in the schools.
4. Analyze your own mathematical strengths and weaknesses as you review the content of junior and senior high school mathematics courses. Remove your own deficiencies before you start to teach.
5. Visit as many classes in mathematics as possible. In addition to observing mathematics teaching on the secondary level, visit classes on the elementary level.
6. Talk with an experienced teacher and make notes on his suggestions.
7. Compare your number symbols with the recommended form shown on a standard chart. Practice writing the correct form of numerals on the chalkboard.
8. Start collecting, classifying, and filing pictures, puzzles, games, and exhibits that relate to the units or topics you may teach.
9. Preplan the semester's work so that proper time allotments may be made for each unit or topic.
10. Read, and if possible, subscribe to such professional journals as the *Clearinghouse*, THE MATHEMATICS TEACHER, *The Arithmetic Teacher*.
11. Review the yearbooks of The National Council of Teachers of Mathematics.

To help you chart a straight course

12. Read the preface or introductory statements in the textbook. Determine the author's viewpoint.
13. Familiarize yourself with the teacher's manual for the textbook.
14. Look upon the textbook as one resource. Avail yourself of others.
15. Remind yourself that it is not the purpose of the mathematics teacher to make mathematicians of all students.
16. Do not limit the mathematics you teach to the immediate demands of the average person.

17. Keep your eye on the distinct purposes of mathematics instruction for the junior high school.
18. Aim for the achievement of a simple, definite, sequential mathematical objective in each lesson.
19. Let the students know what your objectives are for each lesson.
20. Teach students mathematics; do not just teach mathematics.
21. Point out for some students the "bread-and-butter" value of mathematics.
22. Think through and be prepared to answer the students' question, "Why study mathematics?"
23. Crystallize clearly what you believe about transfer of learning as it relates to mathematics.

To motivate or interest students in mathematics

24. Appraise the total classroom atmosphere and quality of human relationships. Study the group rapport.
25. Engender readiness for instruction in each new concept before it is taught.
26. Inventory students' mathematical abilities and skills. They cannot learn something they already know.
27. Take advantage of the wonder motive and the systematizing motive for the learning of mathematics.
28. Preview the lesson for your class. Summarize at the end.
29. Lecture sparingly. Demonstrate copiously.
30. Tap the multitude of number problems suggested by a student's visit to the supermarket.
31. Use intrinsic motivation rather than grades, fear, punishment, or sarcasm.
32. Encourage the enjoyment and thrill of reading about our number system and its development.
33. Refer the students to the daily newspaper for challenging problems in computation.
34. Utilize mail-order catalogues for additional problems.
35. Provide opportunity for gifted students to demonstrate before the class their competence in working with numbers.
36. Cause the students to face and live with unsolved problems. Do not undermine their learnings by giving them help too soon, or too late.
37. Prove to the students that you want them to grow in mathematical insights and understandings.
38. Move toward laboratory teaching in mathematics—an emerging concept.
39. Stimulate interest in mathematics by bulletin board displays and pictures of famous mathematicians.

On materials and other resources

40. Take advantage of a variety of audio-visual resources.
41. Demonstrate with such resource materials as flannel boards with fractional parts, place-value charts for decimals, and hundred boards for per cent.
42. Construct, with your students, an arithmetic kit with fractional cutouts, squared and rectangular strips, compass, ruler, and protractor.
43. Equip your classroom as a laboratory for teaching mathematics; include exploratory, symbolic, and visual materials.
44. See the *22nd Yearbook* of The National Council of Teachers of Mathematics for a list of materials and facilities needed for laboratory teaching in mathematics.
45. Utilize the chalkboard in drills or student demonstrations.
46. Chalk as you talk.

On teaching the lesson

47. Treat the textbook kindly. It is your best friend, but not your master.
48. Calculate the answers. Do not vote or accept majority opinion on problems.
49. Teach in order to achieve pupil participation and discovery.
50. Change the pace in your lessons. Mo-

notony thrives in abstractions.

51. Require an understanding of the process instead of just a correct numerical answer.
52. Leave the "I want" out of your assignments and discussion.
53. Require mathematical effort from all students—gifted, average, and slow.
54. Set high standards of neatness and in the systematic setting down of mathematical computations.
55. Involve students in the core of mathematical learning experiences.
56. Promulgate the correct reading of mathematical symbols.
57. Harmonize orderly and systematic class procedures with the orderliness and exactness of numbers.
58. Realize that teaching pure mathematics to youth may be too rich for consumption for some of them. If you cannot handle an experience unit, try a subject-matter unit.
59. Think in terms of informational mathematics as well as computational mathematics.
60. Find a basis in the previous experience of the student that will involve the problem to be presented.
61. Be cautious concerning student verbalizations. They may not know as much as you think they do.
62. Emphasize the development of the ability to grasp ideas, processes, and principles in the solution of concrete problems, rather than just skill in manipulation.
63. Help the students to develop methods of attack on problems and the ability to see relationships.
64. Guide students in successful procedures for attacking thought problems involving mathematical calculations.
65. Spend precious class time building meanings in number relationships. Collect and record grades and other clerical details before or after class.
66. Supervise your students in the mathematics study period in order to give them individual help.
67. Administer frequent short tests to keep contact with the student's understandings.
68. Test—teach—retest—reteach.
69. Consider as possible causes of student failure: physical defects, poor attitude, and lack of understanding of basic concepts.
70. Diagnose student errors by means of homework, class responses, and diagnostic tests.
71. Be familiar with the school-wide testing program participated in by your students. Chart or graph pertinent mathematical test data about them.
72. Help students develop a recognition of reasonable answers.
73. Admit your mathematical errors. Work toward eliminating them.
74. Build understanding before you drill.
75. Vary drill procedures. Vary the doses, time, content, and participants.
76. Introduce novelty in your drill by the use of such devices as "magic squares" and "lattice" work.
77. Be familiar with and use at different times all of the seven different types of lessons in mathematics: inductive development, drill, deductive development, review, examination, appreciation, and conversational.
78. Try the review lesson which summarizes the main features of the work but in a manner different from that previously used.

On the assignment and homework

79. Assist students in their assignments by systematic procedures and individual help.
80. Do not use homework assignments as punishment.
81. Follow up class corrections of homework with a discussion of errors and their causes.
82. Assume that the students will learn their mathematics in class. Homework provides for reinforcement.
83. Be informed of the school policy regarding homework.

84. Practice differentiation in making assignments.

TO HELP MEET INDIVIDUAL DIFFERENCES

85. Try three levels of work in your class: an enriched program for the accelerated group; a core program for the average or normal group; a minimum program for the slow-moving group.
86. Determine the level at which each student is functioning in each of the seven different areas of arithmetic.
87. Provide for individual differences, since good teaching separates individuals.
88. Practice student grouping in mathematics.
89. Accept the student on the level at which he can perform.
90. Remember that you teach and that they learn—in their own way and at their own rate.

Letters to the editor

Dear Editor:

Here is a problem which might be of interest to your readers. I have never seen it in print before.

Given the integers 1, 2, 3, and 4, express all the integers from 1 through 100 as various combinations of these integers and well-understood mathematical symbols. Each integer may be used only once in each expression. It is not required to use them all in any expression. It is agreed (arbitrarily) to express an integer like forty-two in some other way than 42, mainly to make the problem more interesting. The test for legality of any expression is that it must be clearly understandable by any mathematician who looks at it. Thus ninety could not be expressed as arc sine 1 since some would interpret this as pi over 2. It would be legal, however, to express eighty as 4! over .3.
Examples:

$$63 = 4^3 - 1$$

$$5 = \frac{2}{.4}$$

$$72 = 4! \cdot 3$$

We are not agreed as to whether anti log 2 is a legal representation of 100. If we allow this, we have all the integers from 1 to 100. It seems right to me that if a base is not given we are to assume that the base is 10.

A further question. Is there any number which cannot be produced this way, and what is the smallest such number?

It is clear that there is no largest such number in the set of integers. Surely one could keep on taking factorials as long as strength permits! viz $[[4!]!]!$ . . . .

Sincerely,
DONALD C. DUNCAN
Milton Academy
Milton, Massachusetts

Dear Editor:

The article entitled "An Efficient Way of Factoring the Quadratic Polynomial," in the October, 1960, issue of THE MATHEMATICS TEACHER has prompted me to explain a method I have been using successfully in my algebra classes at Vare Junior High School and Standard Evening High School.

The quadratic polynomial is the result of the following multiplications.

(1) $(ax+b)(cx+d) = acx^2+bcx+adx+bd$
(2) $(ax-b)(cx-d) = acx^2-bcx-adx+bd$
(3) $(ax-b)(cx+d) = acx^2-bcx+adx-bd$
(4) $(ax+b)(cx-d) = acx^2+bcx-adx-bd$

The products of this group yield the following factors by removing a common binomial factor.

$$acx^2+bcx+adx+bd$$
$$cx(ax+b)+d(ax+b)$$
$$(cx+d)(ax+b)$$

The above observations suggest the conversion of a quadratic trinomial into a polynomial of four terms: the product of the factors equal to *abcd* and the sum of the factors equal to *bc* and *ad*. (Note that *bc* and *ad* are factors of *abcd*.)

Examples:

1. x^2+x-56
   $x^2+(+8x-7x)-56$
   $x(x+8)^1-7(x+8)$
   $(x-7)(x+8)$

   The factors of *abcd* (56) are
   1×56
   2×28
   4×14
   (7×8)

2. $12x^2+17x+6$
   $12x^2+(+8x+9x)+6$
   $4x(3x+2)+3(3x+2)$
   $(4x+3)(3x+2)$

   The factors of *abcd* (72) are
   1×72
   2×36
   3×24
   4×18
   6×12
   (8×9)

3. $2x^2-11x+15$
   $2x^2+(-6x-5x)+15$
   $2x(x-3)-5(x-3)$
   $(2x-5)(x-3)$

   The factors of *abcd* (30) are
   1×30
   2×15
   3×10
   (5×6)

(continued on page 165)

• EXPERIMENTAL PROGRAMS

Edited by Eugene Nichols, Florida State University, Tallahassee, Florida

NSF-FSU math campers: evaluations of and by campers

by Williard H. Nelson, Marion F. Smith, Eugene D. Nichols, and Wallace A. Kennedy, Florida State University, Tallahassee, Florida

Thirty of the highest-scoring applicants from over the country were selected for the Third Summer Mathematics Camp which met in 1960 at Florida State University. The six-week 1958 and 1959 institutes sponsored by the university and the National Science Foundation have been described by their director, Dr. Eugene Nichols.[1] Compelling opportunity was afforded to study a group of very talented high school youth. A year ago an initial study of the campers was conducted.[2] Employing the 1960 group, Kennedy, *et al.*, evaluated the Third Revision of the Stanford-Binet Intelligence Scale as a test for individuals at the upper levels of intelligence.[3]

There were reasons of expedience as well as long-range, theoretical reasons for obtaining information concerning reactions of these very special students to each other, to content offerings, and to other aspects of the camp. The camp is intended to pioneer; it began when there was but a handful of such institutes for high school youth. Hence, both for provision of new mathematical experience for the participants and also for suggesting desirable content for institutes elsewhere, the campers' evaluations of the camp could be very useful as well as interesting. The campers' perceptions of the camp as related to motivation and facilitation of learning have potentially significant implications for other instruction of talented groups.

With three exceptions who had not reached sixteen, the thirty participants were sixteen- and seventeen-year-olds, and all but the lone tenth grader had one year of high school remaining. They had been selected upon recommendation by their schools on the basis of performance on the Mathematical Association of America and the Society of Actuaries' High School Mathematics Contest examinations. These twenty-four boys and six girls represented all sections of the nation except the Northwest.

Verification of high mathematical ability was obtained from administration of the Mathematics Form of the Sequential Tests of Educational Progress; the mean was 85.9, with a S.D. of 8.4. That this mode of selection brought to the top a

[1] Eugene D. Nichols, "A Summer Mathematics Camp for Talented High School Students," *The Bulletin of the National Association of Secondary High School Principals*, XLIII (May, 1959), 100–103; "A Summer Mathematics Program for the Mathematically Talented," THE MATHEMATICS TEACHER, LIII (April, 1960), 235–240.

[2] Wallace A. Kennedy and the Human Development Clinic Staff, "A Multidimensional Study of Mathematically Talented Adolescents, *Child Development Publications*, December, 1960.

[3] Wallace A. Kennedy, W. H. Nelson, R. Lindner, H. Moon, and J. Turner, "Psychological Measurements of Future Scientists," *Psychological Reports*, 1960, 7, 515–517, in press.

group whose general intelligence is high was verified; means of 136.8 on the verbal scale of the Wechsler Adult Intelligence Scale and of 137.4 on the L-M Binet scale were obtained.[4]

Instructional program

The instructional program, in addition to the participants' independent work, consisted of five hours of daily class instruction. The following courses were offered:

Sets, relations, and elementary functions. This course included an elementary development of set theory, study of Cartesian products, relations, functions—in particular, polynomial, exponential, and log functions.

Introduction to matrix algebra. This course presented a study of matrices using the textbook *Introduction to Matrix Algebra* by the School Mathematics Study Group. Topics considered were basic operations with matrices, the algebra of two-by-two matrices (fields and rings), inversion of matrices, the determinant function, isomorphism between complex numbers and matrices, applications of matrices to the solution of systems of equations, the algebra of vectors, vector spaces and subspaces, invariant subspaces, and rotations and reflections.

This course was used for observation by the nineteen superior mathematics teachers participating in the National Science Foundation Supported Teacher Institute. Daily observation was followed by a seminar devoted to deeper study of matrix algebra and elements of set theory.

Theory of numbers. This course took up the study of prime numbers, unique factorization, congruencies, elementary theory of groups, integral domains, fields, and the algebra of polynomials.

Programming techniques for IBM 650. Topics undertaken were the study of the memory, the arithmetic unit, and the control unit, coding (arithmetical shift, branch, and Watson System operations), programming (including the use of loops), floating point calculations, double precision calculations, tracing, Newton's iterative method, solution of simultaneous equations. In connection with this course the following problems were handled by students as part of their independent research: multiplication and inversion of matrices as a means of solving simultaneous equations, composition of music on the computer, calculation of Fibonacci numbers to twenty digits, calculation of probability at roulette, computation of square roots and cube roots to ten places, computation of e, an attempt to prove the infinity of twin primes, calculations of distances between nearby stars, computation of mass, density, and period of revolution for planets.

Combinatorial topology. Here students concentrated upon a study of real n-tuples, Euclidean spaces, linear dependence, semilinear maps, chains and chain groups, spheres and tori, projective spaces, and homomorphisms.

Introduction to Russian. The course covered a study of grammar, vocabulary, and reading and pronunciation practices. Also included was introduction of such materials as Russian newspapers, folk songs, and Russian language recordings. One three-hour lecture on the history of Russia was given.

The camp program also included a number of special lectures by visiting mathematicians. These lectures covered such topics as probability, simple harmonic motion, Fibonacci series and matrices, and transfinite numbers.

Method and procedure

Subjects. Twenty-eight of the thirty campers were available for this study.

Procedure. A special questionnaire was prepared to assess reactions to the special situation with its academic and other offerings. On the day before the camp ended, the questionnaire was administered by an instructor who had demonstrated his abil-

[4] Kennedy, *et al.*, 1959, 1960, *op. cit.*

ity to obtain especially frank responses. Two of the authors had held earlier discussions with the participants and could thus compare questionnaire responses with oral statements. The questionnaire was focused especially upon reactions to instruction and living arrangements, including recreation, and upon vocational and academic plans. A sociometric questionnaire had been administered in May to all students in homerooms of the selectees; a similar device was administered to the campers in September.

Data from questionnaires

Duplications. Except in the case of set theory, one major intent of the camp was fulfilled: the topical content of instruction generally avoided duplication of the content of high school instruction which the campers had already experienced. However, only topology, calculus, and number theory had not been dealt with in the classes of one or more campers, even though nine students reported no overlap at all. These findings suggest that (1) changing mathematical conceptions and methodology are beginning to have their impact, at least in certain schools, and (2) some schools have felt little, if any, such impact.

With the exception of sets and matrix algebra, the courses at the camp had not been subjects which the students had pursued independently. Each of these topics had been studied independently by fewer than one-fourth of the campers. It places the matter in another light, however, to note that at least half of the campers had studied at least one topic without the direction and assignment afforded by course work. This appears to contrast with the findings of Gallagher, who studied elementary pupils above 150 I.Q.[5] Gallagher has pointed out the disappointingly small extent to which independent pursuit of mathematics characterizes even the extremely bright individual, at least in the elementary-school years.

Such findings suggest the need for provocatively written materials on a wide variety of mathematical topics and at varying levels of sophistication. Materials need to be of such levels of information loading, vocabulary, etc., as to permit and encourage independent access to new ideas by interested students at many levels of readiness.

Plans for the next year of high school. The students each planned to take an average of nearly two mathematics courses during their final year of high school. These plans were set prior to their arrival at the institute. Courses which will be taken apparently will be determined principally by offerings of the individual high schools. Thus calculus and elementary analysis, for example, might be taken by more of the campers were these subjects more widely available in the local schools.

Liking for course material. Distinct differences in attitude toward subjects were expressed. This occurred both in questionnaire ranking of and verbalization about courses. Because of the confounding of attitude toward content with attitude toward instructor, and the further factor of marked differences in student readiness for the material, little will be said here relative to attitude toward topics. As Table 1 indicates, topology was among the most preferred but was also among the least liked. On the basis of questionnaire responses and oral expressions, this ambivalence appears to be a function of several variables, including mathematical aspirations or their lack. Some material in certain of the courses was clearly more difficult than students had heretofore encountered. Bright though the students were, and excellent though their backgrounds and knowledge appeared to be, some of the materials proved to be beyond their immediate comprehension. In the face of such difficulty they displayed a

[5] James J. Gallagher and Thora Crowder, "The Adjustment of Gifted Children in the Regular Classroom," *Exceptional Children*, XXIII (April, 1957), 306–12, 317–19; "The Adjustment of Gifted Children in the Regular Classroom: Case Studies," *Exceptional Children*, XXIII (May, 1957), 353–63, 396, 398.

TABLE 1

MOST AND LEAST PREFERRED TOPICS OFFERED TO MATH CAMPERS

Subject matter	Most liked	Least liked
Matrix algebra	18	1
Topology	16	9
Theory of functions	6	8
Non-Euclidean geometry	3	0
Theory of equations	3	0
Russian	3	1
Calculus	2	2
Number theory	2	16
Probability theory	2	0
Set theory	2	5
Symbolic logic	1	0
Trigonometry	0	5
None	1	6

progressive dissociation from the course, failing to take the necessary steps to catch up. In this respect, they resembled a heterogeneous or "average" group of students. In the judgment of the writers, difficulty level is a factor which must be carefully taken into account in any attempt to work with especially talented youth.

Visiting lecturers. In general, the participants liked the visiting lecturers. There were some dissenting notes, most relating to lectures beyond the comprehension of the students or too unrelated to the materials being studied. Even with a considerable amount of preliminary work to assure adequate relation between materials presented by special lectures and the content in class at the given time, some undesirable gaps are likely to occur.

Russian. The majority of the campers expressed appreciation for the inclusion of Russian language in their curriculum. Most of them believed that this language would be useful to them in the future. A majority planned to pursue it further; four students intended to continue such study independently. Furthermore, three will have the opportunity to study Russian in their high schools during the coming year. Of six students who believed it of little value, three plan to continue studying Russian!

Daily schedule. Three-fourths of the students indicated liking for their daily schedule. Eight of the campers disliked the schedule, and two expressed no concern. Highly though Russian was regarded, its placement as the fifth hour of classwork gave rise to the belief that a different arrangement might have been preferred. The feeling was expressed that mathematics courses would be less satiating if Russian were interposed. Hence, the participants demonstrated some awareness of the conditions under which effective learning takes place.

Recreational facilities. The planned recreational programs and facilities were generally rated favorably, but numerous specific complaints were registered. Most of the complaints point to the characteristics which talented youngsters possess in common with other adolescents. Specifically, the campers wanted greater access to the swimming pools, canoes, skating, athletic equipment, and—not so characteristic of adolescents—to the IBM 650 computer, which had not been expected to come under the heading of recreation. A source of disappointment was that there was so little to do that was obviously inviting on the week ends. Generally, planned recreation was provided on either Saturday or Sunday, but on one of these days the campers were left to their own resources. Because of the motivational importance of this phase of their living during a fairly protracted period, physical recreation and general socialization appear to need even greater attention than previously was given.

Dormitory regulations. Certain dormitory regulations provoked vociferous complaint. The participants were nearly unanimous in their objections to the early curfew which required that they be in the dormitory at 8 o'clock and have their lights out at 10 o'clock. Much less restricted at home, they were accustomed to reading in bed. As a group, the students believed themselves to have sufficiently important purposes and to be of sufficient

maturity to warrant greater control over their own lives. Their complaints are the more readily understood in light of the fact that they shared quarters with a less intellectually select group of lower chronological age, the FSU Music Campers, who had been selected on the basis of interest in music. The latter youngsters behaved in ways that often prevented intense concentration and study on the part of the mathematics students. This resulted in a strong plea for separate housing, as well as a desire for a more centralized location to facilitate class attendance and access to the cafeteria. In general, such a mature group of adolescents need recognition for the implications of their maturity.

Psychological investigations

Because the campers comprised such a unique and original sampling, many investigators wanted to study them in special ways. In a sense, it could be seen as an obligation of these specially gifted and benefited youngsters to serve as subjects so that wiser provisions for the development of other such youngsters may be made in the future. However, the desired investigations were necessarily restricted to a small number.

To determine the reactions of the mathematics campers to the psychological investigations that were conducted, the questionnaire approach and the opportunity to express feelings orally in subgroup meetings were used. Responses revealed some sense of interference with the program and some mild annoyance. At the same time, the campers were interested in the results and explicitly requested that feedback be provided. Oral expressions were generally to the effect that although the campers made fun of the questionnaires and the examiners, they were actually very much interested in the investigations.

Attitude toward mathematics. Campers were asked to report any changes that might have come about in their attitudes toward mathematics which could be ascribed to the summer institute. About one-fourth of the group reported no change. But it is interesting to note that more than half expressed a perception of the tremendous range and scope of the field; they had not previously been aware of the magnitude of the field of mathematics. A nearly equal number of the participants had begun to question their own mathematical ability; the questions were of such intensity as to suggest anxiety of a sort which few of the campers had previously experienced, at least in the academic sphere. This was largely a result of finding themselves in competition, for the first time, with a group of students of similar mentality. This competition had a variable but generally positive effect. Five campers felt increased determination to succeed and the same number developed even greater interest in mathematics. Two reported waning interest, with an equal number expressing the view that the institute had helped to crystalize their interests so that selecting a future career would be simplified. One revealed a shift in areas of interest within the field.

Other responses to the institute. About two-thirds of the participants expressed their enjoyment of the camp in direct words. This figure may be misleadingly low because tabulations were made only when the expression was direct rather than implied. In spite of the specific complaints of the sort already discussed, positive reactions to the institute, in general, could be inferred. Some 60 per cent found the camp an opportunity for intellectual stimulation; about 40 per cent experienced a worthwhile sense of social stimulation, and an equal number saw values in their experiences as an introduction to college living. More than one-third suggested that more time be provided for instruction in the use of the IBM 650 computer. One-sixth revealed an increased sense of maturity and independence gathered from "being on my own" in a university center. Last, but far from least, another sixth showed a sense of increased understanding

TABLE 2

CAMPERS' FIRST AND SECOND VOCATIONAL PREFERENCES

Vocation	First choice	Second choice
Mathematician	20	5
Physicist	2	8
Astronomer	1	3
Chemist	0	3
Chemical engineer	0	1
Electronics engineer	1	0
Attorney	1	0
Biologist	1	0
Physician	1	0
Teacher, foreign languages	1	0
Architect	0	1
Dentist	0	1
Housewife	0	1
Philosopher	0	1
Politician	0	2
Psychologist	0	1

of others—with specific reference to how less able students must feel in class with brighter ones. These findings are consistent with those derived somewhat more subjectively from the one-page essays written by the campers of the two previous years.

FUTURE VOCATIONS

First and second vocational choices. Table 2 reveals the mathematics campers' first two choices of vocation. All of their initial selections are at the professional level, with all but one of the second choices at this level. More than two-thirds of the participants expressed aspirations to become mathematicians, if a broad definition of the term is accepted. The first choices encompass a total of eight vocations, of which only two are not in the general realm of science and mathematics. If one includes vocations demanding considerable sophistication in mathematical procedures, 24 of the 28 students have such preferences as first choice. Physics was second as the prime vocational objective, but first as the second-choice objective.

Temporal duration of participants' first vocational preference. Eleven of the campers settled upon their vocational objectives near the end of junior high school or at the beginning of high school. Nine selected their future careers within the past year. Three had had specific vocational aims since elementary school, and five had held such aims as long as they could remember. Therefore, they not only elect high-level occupations, but also tend to make selections early and adhere to them over a relatively long period of time.

Degree of certainty regarding entry into chosen occupation. More than half of the students were reasonably certain that they would actually enter their chosen fields. Only three professed to be absolutely certain, and nine expressed themselves as somewhat uncertain. One student was very unsure, and another entirely uncertain. Thus, they seem both optimistic and realistic toward future aspirations.

SUMMARY

The purpose of this article is to report reactions to a six-week summer institute in mathematics for mathematically-gifted high school students from all over the nation. With few exceptions, these highly talented youths were in the upper percentile on individual intelligence tests. They appeared on the basis of questionnaire and essay responses to be both similar to and different from other adolescents. Behaving much like others, they showed appreciation for efforts made for them, disliked vigorously certain limitations on their independence, liked socialization, and were at least partially lost when material presented was more difficult than they were ready for. They were unlike the generality of adolescents in the certainty and professional level of occupational choice and the way in which they wanted to spend significant portions of their free time—specifically, on such things as the 650 computer and sedentary intellectual pursuits.

The changing course content, in at least some schools, and the independent pursuit of mathematical topics by talented individuals dictate the continued reappraisal of curricular materials for such special institutes as the FSU Camp.

• HISTORICALLY SPEAKING,—

Edited by Howard Eves, University of Maine, Orono, Maine

Axioms in algebra—where did they come from?[1]

by John K. Baumgart, Manchester College, North Manchester, Indiana

Today we hear a great deal about revision of the mathematics curriculum on both the high school and college levels. At the high school level some proposals involve the presentation of algebra from a more axiomatic viewpoint—in much the same way as geometry has been built logically from a set of axioms. It is interesting and instructive to trace the development of the axioms for algebra.

Every mathematics teacher has seen in at least one algebra book a set of "rules" such as

$$a+b=b+a,$$

$$ab=ba,$$

$$a(b+c)=ab+ac,$$

and so on. Sometimes the reaction has been, "Why bother to list such obvious statements?"

We shall try to show that these "obvious statements" were abstracted from a system of arithmetic and algebra (and, in fact, from geometry too) which just grew, like Topsy. But the system grew to fulfill a need for that kind of mathematical structure which it eventually became; this structure is now called a *field*.

Consider, for example, the property $a+b=b+a$. In rudimentary and concrete terms this was obvious even to primitive man. He knew that it made no difference whether he first made three vertical marks on a wall and then two, or first two and then three. This was the intuitive level of using the commutative law: $3+2=2+3$.

The Egyptians (as early as 1850 B.C.) showed a curious awareness that something was involved in assuming that $ab=ba$. A. B. Chace in his commentary on the Rhind Papyrus remarks that the Egyptian method of multiplication emphasized a distinction between multiplicand and multiplier, but it was known that if the two were interchanged the product would be the same [1].[2] An example of this interchange of factors occurs in Problem 26 of the Rhind Papyrus [2]. The problem has additional interest as an example of the Egyptian method of solving what we would today call an algebra problem. The method used is known as the method of *false position;* that is, a solution is guessed at and then revised.

It seems worth while to give the problem just as it appears in the Rhind Papyrus, which includes the statement of the problem, its solution, and a verification. (The English translation uses modern notation for numbers.)

[1] Written while on an NSF Science Faculty Fellowship at the University of Michigan. Many helpful suggestions were made by Professor Phillip S. Jones.

[2] Numbers in brackets refer to the references given at the end of the paper.

Problem 26

A quantity and its $\frac{1}{4}$ added together become 15. What is the quantity? Assume 4.

√	1	4
√	$\frac{1}{4}$	1
total		5.

As many times as 5 must be multiplied to give 15, so many times 4 must be multiplied to give the required number. Multiply 5 so as to get 15.

√	1	5
√	2	10
total	3.	

Multiply 3 by 4.

	1	3
	2	6
√	4	12;

the quantity is		12
	$\frac{1}{4}$	3
	total	15.

Notice in the above problem that the Egyptian scribe determines that 4 must be multiplied by 3. Then to get the product he multiplies 3 by 4, the reverse order. The probable reason for this is that the Egyptian did all his multiplication by doubling; consequently it was easier to multiply by powers of 2 than by other numbers. Actually to multiply 4 by 3, the Egyptian would have to double *and* add:

√	1	4
√	2	8
totals	3	12.

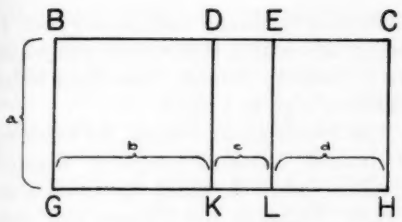
The Egyptians also freely used the distributive law, $a(b+c)=ab+ac$, but apparently without recognizing anything basic. An illustration of this occurs in Problem 68 of the Rhind Papyrus [3]. To double the number 3 21/64, which the scribe writes as 3+1/4+1/16+1/64, he simply doubles each term and gets 6+1/2+1/8+1/32, which, of course, is equal to 6 21/32. Note the use of *unit fractions* (fractions with unit numerators), which was standard procedure.

The ancient Babylonians (*ca.* 1700 B.C.) also used the commutative and distributive laws. These laws were tacitly assumed in their rhetorical algebra when, in effect, they used such formulas as $(a+b)^2=a^2+2ab+b^2$. Van der Waerden gives some excellent examples of Babylonian algebra where these ideas are implied [4]. He writes:

> The formulas
>
> (11) $\qquad (a+b)^2=a^2+2ab+b^2$
>
> and
>
> (12) $\qquad (a-b)^2=a^2-2ab+b^2$
>
> must also have been known to the Babylonians. For the old-Babylonian text BM 13901 contains the following problem: "I have added the areas of my two squares: 25,25.[3] (The side of) the second square is $\frac{2}{3}$ of that of the first plus 5 GAR."
>
> That is to say:
>
> $$x^2+y^2=25,\ 25,$$
> $$y=(2/3)x+5.$$
>
> In order to substitute the value of y, obtained from the second equation, in the first equation, formula (11) has to be used:
>
> $$(0;\ 40x+5)^2=0;\ 40^2x^2+2\cdot 0;\ 40\cdot 5x+5^2.$$
>
> This leads to a quadratic equation
>
> $$ax^2+2bx=c$$
>
> for x, in which
>
> $$a=1+0;\ 40^2=1;\ 26,\ 40,$$
> $$b=5\cdot 0;\ 40=3;\ 20,$$
> $$c=25,\ 25-5^2=25,\ 0.$$
>
> The text first calculates the three coefficients a, b, c; then the quadratic equation is solved by use of the correct formula:
>
> $$x=a^{-1}(\sqrt{ac+b^2}-b),$$
>
> and finally $y=(\frac{2}{3})x+5$ is determined. It follows that the method of elimination, described above, was used and that the formula (11) was known.

Looking at Greek mathematics, we see that Euclid (320 B.C.) was more aware of the explicit nature of the distributive law. In his geometric algebra of Book II he states and proves Proposition 1: "If there be two straight lines, and one of them be cut into any number of segments whatever, the rectangle contained by the two straight lines is equal to the rectangles

[3] The Babylonian numerals employ 60 for a base. Hence 25,25 means $25\cdot 60+25$. Again, 1;26,40 means $1+26/60+40/60^2$.



contained by the uncut straight line and each of the segments [5]." That is (See the figure above.),

$$\text{area } BGHC = \text{area } BGKD + \text{area } DKLE + \text{area } ELHC,$$

or

$$a(b+c+d) = ab+ac+ad.$$

Although Euclid was quite explicit on the distributive property as applied to areas, we find that in Book VII, where Euclid deals with numbers as such, it does not occur to him to discuss the same distributive property. Perhaps he regarded this property as "only natural" for numbers and felt that no specific mention was necessary.

The question naturally arises as to when mathematicians began to recognize as axioms the properties which had been previously taken for granted.

Of the axioms for arithmetic and algebra, we first want to consider three in particular:

The associative law: $(a+b)+c=a+(b+c)$ and $(ab)c=a(bc)$.

The commutative law: $a+b=b+a$ and $ab=ba$.

The distributive law: $a(b+c)=ab+ac$.

These axioms received the above names during the first sixty years of the nineteenth century [6]. Interestingly enough, the last two were given their present names by Servois in a discussion involving *functions* [7].

Servois comments that if $f\phi z=\phi fz$, where f and ϕ are functions and z an independent variable, then the functions are called *commutative*. He also says that if $f(x+y+\cdots)=fx+fy+\cdots$, then the function is called *distributive*. (He used parentheses more sparingly than we do today.)

The associative law was mentioned by Hamilton in 1853 in his *Lectures on Quaternions*. In the preface he stated, "to this associative property or principle I attach much importance [8]." Of course the associative law was used long before it was named, and it was already explicitly noted in 1830 when Legendre called attention to it in his *Théorie des Nombres*. Legendre wrote, "One ordinarily supposes that in multiplying a given number C by another number N which is the product of two factors A and B, one gets the same result whether he multiplies C by N all at once, or C by A and then by B." Symbolically, Legendre wrote: $C\times\overline{AB}=\overline{CA}\times B$ [9].

It is now only natural to ask, "What are the other axioms which in the past have been tacitly assumed when working with arithmetic and algebra?" To answer this question it is desirable from both historical and logical considerations to talk about a *group*.

A group is defined to be a set of elements (say numbers) and an operation (say addition) with the properties:

1. *Closure* (i.e., the sum of two numbers in the set is also in the set);
2. *Associativity:* $(a+b)+c=a+(b+c)$;
3. *Existence of an identity element*, zero (0), such that $a+0=0+a=a$ for every number a in the set;
4. *Existence of an inverse element* $(-a)$ for each a in the set such that $a+(-a)=(-a)+a=0$.

If we add the property

5. *Commutativity:* $a+b=b+a$, we say the group is a *commutative group*.

For example, the integers (positive, negative, and zero) under ordinary addition form a commutative group since all five of the above axioms hold. But the odd integers under ordinary addition do not form a group, since the closure property does not hold—the sum of two odd integers is *not* an odd integer. Axiom 3 also

fails, since zero is not among the odd integers; and as a result Axiom 4 also fails to hold.

Of course, all five of the above axioms were known and used long before the formal definition of a commutative group was given. Let us look at each of the five axioms more closely.

The first axiom (of closure) was what early mathematicians were really concerned with when they saw that fractions were needed to make division always possible. They had no trouble dividing 12 by 4 or 15 by 3, but to divide 2 by 3 they needed a fraction $\frac{2}{3}$. (The idea of using fractions to represent "parts" seems fairly natural, but a general reluctance to do so is still evident in using, for example, 12 ounces in place of $\frac{3}{4}$ pound.)

The second axiom (of associativity) seems, at first glance, to be the least interesting of the five. One is inclined to say for addition, "Of course $(3+5)+2=3+(5+2)$," or to say for multiplication, "Of course $(3\cdot5)\cdot2=3\cdot(5\cdot2)$." The associative law (for both addition and multiplication) continues to hold as we enlarge the number system from integers to the rational numbers (integers and fractions) and then to the real numbers (the rationals and the irrationals). In fact, the associative law still holds for the complex numbers—numbers like $3+4i$ with one real part and one imaginary part. It even holds for Hamilton's *quaternions*—hypercomplex numbers like $3+2i+5j+4k$ with one real part and three different imaginary parts. But the associative law finally breaks down when multiplying Cayley numbers (one real part and seven imaginary parts).

With respect to the third axiom which states the existence of an identity element, we have already mentioned zero when speaking of *addition* of numbers. The story of zero is a fascinating one. The symbol and concept of zero were achieved in response to several needs. The role of zero in the principle of position is fairly well known. Another role was played in the solution of equations such as $5+x=5$. It was in this setting that a need was sensed for an identity element: something which added to 5 gives 5 again.

The fourth axiom stating the existence of inverse elements, requires negative numbers (when the group operation is addition). Although the Chinese had some understanding of negative numbers (and wrote positive coefficients in red, negative in black), the first mention of negative numbers in an occidental work is by Diophantus (*ca.* 275). He gave "rules" for operating with "forthcomings" and "wantings" in situations where the final result was a "forthcoming," but he apparently had no conception of a negative number in the abstract, and regarded the equation $4x+20=4$ as absurd because it had $x=-4$ as a root [10]. Cardan, in his *Ars Magna* (1545), recognized negative roots and stated laws governing negative numbers. The solution of an equation like $x+7=0$, with a root $x=-7$, was preliminary to stating essentially the same idea in more sophisticated terms: the (additive) inverse of 7 is -7.

The fifth axiom (of commutativity) was, as we have seen, tacitly assumed and used for a long time. Its basic character was perhaps most dramatically revealed when Hamilton discovered that his newly invented quaternions did not necessarily obey the commutative law (for multiplication).

Hamilton's discovery that his new "numbers" (beyond the complex numbers, hence sometimes called hypercomplex numbers), quaternions, did not obey one of the expected properties of numbers should have been enough to raise the question, "What axioms are needed to define various systems?"

But the answer came, in part, in response to a somewhat different line of inquiry. It was in connection with the algebraic solution of equations that the theory of groups was first studied [11]. The pioneers in this undertaking were Lagrange (1770), Ruffini (1799), Cauchy

(1814), Abel (1824), and especially Galois (1831). The first discussion of the theory of groups from an abstract point of view was by Cayley in 1854, and the earliest explicit sets of axioms were given by Kronecker in 1870 and H. Weber in 1882.

It was noted above that the integers form a group if the group operation is addition. But if we had chosen multiplication as the group operation the integers would *not* form a group, since 7, for example, would not have an inverse in the set of integers; that is, Axiom 4 would not hold since it is impossible to find an integer x such that $7 \cdot x = x \cdot 7 = 1$. (Our identity element for multiplication is 1, since $N \cdot 1 = 1 \cdot N = N$.)

But the rational numbers (the integers and the fractions) *do* form a commutative group under multiplication—except for zero, which has no inverse. This is easily seen by checking the five group axioms using nonzero fractions. The rational numbers (zero now included) also form a commutative group under addition. Moreover, these two groups are "connected" by the distributive law: $a \cdot (b+c) = a \cdot b + a \cdot c$, and we say that multiplication is distributive over addition.

Taking inventory of our axioms up to this point we see that we now have eleven axioms.

Additive group:

1. Closure: $a+b$ is in the set.
2. Associativity: $(a+b)+c = a+(b+c)$.
3. Identity element, zero (0), such that $a+0 = 0+a = a$.
4. Inverse element, $(-a)$, such that $a+(-a) = (-a)+a = 0$ for every element a in the set.
5. Commutativity: $a+b = b+a$.

Multiplicative group:

6. Closure: $a \cdot b$ is in the set.
7. Associativity: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
8. Identity element, one (1), such that $a \cdot 1 = 1 \cdot a = a$.
9. Inverse element, a^{-1}, such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$ for every nonzero element a in the set.
10. Commutativity: $a \cdot b = b \cdot a$.

The distributive law:

11. $a \cdot (b+c) = a \cdot b + a \cdot c$.

We say that these eleven axioms determine a *field*. This means, for example, that the rational numbers under ordinary addition and multiplication form a field, since all eleven axioms are satisfied. The real numbers, under addition and multiplication, also constitute a field; so do the complex numbers.

The theory of fields was suggested by Galois and was given a concrete formulation by Dedekind in 1871. The earliest expositions of the theory from the general or abstract point of view were given independently by H. Weber and E. H. Moore in 1893 [12].

As we have seen above, the "natural," or expected, properties of the rational numbers (as summarized by the eleven field axioms) still hold when we enlarge the system to include the irrational numbers. And if we take this larger system (the real number field) and add to it the imaginary numbers, the resulting system (the system of complex numbers) is again a field (the complex number field).

Can we continue this process? Are there larger number fields than the complex number field? The answer is *no*; it is not possible to extend the complex number field to a larger field. Both Hamilton's and Cayley's hypercomplex numbers fail to satisfy all of the field axioms, as was pointed out above.

References

1. A. B. Chace, *The Rhind Mathematical Papyrus* (Oberlin: Mathematical Association of America, 1927), I, 6.
2. *Ibid.*, p. 68.
3. *Ibid.*, p. 104.
4. B. L. Van der Waerden, *Science Awakening* (Groningen, Holland: P. Noordhoff, Ltd., 1954), pp. 68–69.
5. T. L. Heath, *The Thirteen Books of Euclid's Elements* (Cambridge, England: University Press, 1908), I, 375.

6. G. A. Miller, "A Third Lesson in the History of Mathematics," *National Mathematics Magazine*, 15 (1940–1941), 234–244.
7. F. Servois, "Essai sur les principes du calcul différentiel," *Annales de Mathématiques Pures et Appliquées*, 5 (1814), 98.
8. W. R. Hamilton, *Lectures on Quaternions* (Dublin: Hodges and Smith, 1853), p. 61 of preface.
9. A. M. Legendre, *Théorie des Nombres* (Paris: Librairie Scientifique A. Hermann, 1900), I, 2.
10. D. E. Smith, *History of Mathematics* (New York: Ginn and Co., 1953), II, 257–259.
11. E. V. Huntington, "Note on the Definitions of Abstract Groups and Fields by Sets of Independent Postulates," *Transactions American Mathematical Society*, 6 (1905), 181.
12. *Ibid.*, p. 183.

The Cardano-Tartaglia dispute

by Richard W. Feldmann, Jr., University of Buffalo, Buffalo, New York

Introduction

As the Middle Ages came to a close, a rebirth of scientific inquiry occurred. Most large cities had universities, but these consisted primarily of lecturers, as there were few books beyond those of the classical authors. Hence the reputation of a university depended upon its ability to provide the best lecturers. But who was the best lecturer? One way to settle this was in a public debate with the winner gaining prestige and academic positions while the loser was ignored.

Since a scholar had to outperform all challengers to maintain his position, he needed every trick he knew. If someone knew something that no one else knew, his fame was insured. One assurance of success would be a knowledge of the general solution to the previously unsolved cubic equation. It was the search for this solution that led to the dispute between Gerolamo Cardano and Niccolo Tartaglia. But it is impossible to discuss the debate without first giving brief biographies of the antagonists to show why they acted as they did.

The life of Tartaglia

Niccolo Tartaglia—his actual family name was Fontana—was born in Brescia, Italy, about the turn of the sixteenth century.[1] When the French sacked Brescia in 1512, the youthful Fontana suffered severe saber cuts about the face and mouth which caused a speech impediment. Because of this he was nicknamed Tartaglia, "the stutterer." He later wore a long beard to hide the scars, but he could never overcome the stuttering.

His education was very meager. In fact he tells, in one of his books, that his mother had accumulated a small amount of money so that he might be tutored by a writing master. The money ran out, so he says, when the instruction reached the letter *K* and his education stopped before he could write his own initials. But being an enterprising youth, he stole the master's lecture notes and completed the course on his own. In times of extreme poverty he even used tombstones in place of writing slates.

His mathematical knowledge was completely self-taught, but his attempts to teach himself Latin resulted in failure and he was forced to write his treatises in Italian instead of the accepted Latin. He taught primarily in Venice, and, once es-

[1] The actual year is in doubt; Oystein Ore lists it as 1499 in *Cardano, the Gambling Scholar*, and D. E. Smith gives 1506 in his *History of Mathematics*.

tablished, lived quite comfortably on his wages and the wagers he won in public disputes and challenges.

The life of Cardano

The other participant, Gerolamo Cardano,[2] was born in Milan in 1501. His father, a lawyer and lecturer in geometry, married Cardano's mother, who came from a "socially unacceptable" family, a few years after Cardano's birth. This led to later complications.

He started his studies at the University of Pavia, but transferred to the University of Padua when war broke out. At college he supplemented his meager finances by constant gambling, at which he became very proficient. He wrote a treatise, *Liber de ludo aleae*,[3] which not only introduced the idea of probability as we use it today, but also included ways to cheat in these games.

At twenty-five, he graduated as a doctor of his chosen profession, medicine. Due to his illegitimate birth, he was not allowed to practice in Milan. Later he was recognized as a doctor and proved his ability by becoming the second most renowned medical expert in Europe at that time. Most of his 412 written works[4] were in the field of medicine, popular science, and astrology. One of these was an autobiography in which he describes himself as follows: "living from day to day, outspoken, despising religion, mindful of injuries caused by others, envious, melancholy, a spy, a betrayer of trusts, a deviner, a caster of charms, subject to frequent failures, hateful, given to shameful lust, solitary, unpleasant, strict, foretelling the future willingly, jealous, obscene, lascivious, abusive in speech, inconsistent, two-faced, dishonest, a slanderer, entirely unparalleled in vices, and by nature incompatible even with those with whom he converses daily."[5]

His sons were "chips off the old block," as one robbed his father and the other married and later murdered an immoral girl.

In 1570, Cardano was arrested and jailed on a charge of heresy. The charge could be substantiated by a horoscope of Christ, and by a book which Cardano had published praising Nero, the Roman emperor well known for his persecution of the early Christians. After he was convicted, he was forced to admit and renounce his heresies. As a punishment he was denied the right to lecture publicly and was ordered to refrain from writing and publishing books. Heartbroken, he accompanied one of his students to Rome, where he received an invitation to become a consultant to the College of Physicians. A pension was received from the Pope and soon, possibly insane,[6] he died.

A student of Cardano's who received fame as a mathematician was Ludovico Ferrari. He entered Cardano's house as a servant, soon was elevated to secretary, and became a public lecturer before he was twenty years old. His major contribution to mathematics was a general solution to the biquadratic equation $x^4+ax^2+b=cx$. Unfortunately his career was cut short by premature death when in his early forties.

The dispute[7]

About 1510, Scipione del Ferro found a general solution to $x^3+ax=b$, but he died before he could publish his discovery. His student, Antonio Maria Fiore, knew the solution and attempted to gain a reputa-

[2] Hieronymus Cardanus in Latin; also translated into English as Jerome Cardan.

[3] *A Book on Games of Chance*

[4] At the time of his death, 131 of his works had been published, 111 existed in manuscript, and he claimed to have burned 170 which he found unsatisfactory.

[5] A quote taken from the introduction by Gabriel Naude in Cardano's autobiography, *Liber de propria vita*, Amsterdam, 1654.

[6] Charles W. Burr, *Jerome Cardan as Seen by an Alienist*, in University of Pennsylvania: University Lectures Delivered by Members of the Faculty in the Free Public Lecture Course, 1916–1917, vol. 4.

[7] The two main sources for the dispute are Oystein Ore, *Cardano, the Gambling Scholar*, Princeton: (Princeton University Press, 1953) and M. A. Nordgaard, "Sidelights on the Cardan-Tartaglia Dispute," *National Mathematics Magazine*, XII (1937), 327–346.

tion by exploiting his master's discovery. He challenged Tartaglia with thirty questions, all of which reduced to the solution of $x^3+ax=b$. Tartaglia had the general solution to $x^3+ax^2=b$, so he responded with thirty questions of a more general theoretical nature, although some resolved to this equation. Besides the prestige to be gained, the winner and his friends were to receive thirty banquets from the loser. Just before the time limit elapsed, Tartaglia found general solutions to both $x^3+ax=b$ and $x^3=ax+b$.[8] With these, Tartaglia solved all of Fiore's problems, but Fiore was unable to solve any of the questions proposed by Tartaglia and so was vanquished. The banquets were not collected.

At this time Cardano was writing the *Practica arithmeticæ generalis*, which would encompass, arithmetic, geometry, and algebra. Inasmuch as Fra Luca Pacioli had earlier stated that there could not be a general solution to the cubic, Cardano had ignored this topic. Upon hearing that Tartaglia had a solution for $x^3+ax=b$, he tried to find one. Failing, he asked Tartaglia for the solution so that he might publish it in a special section of the *Practica arithmeticæ generalis* under Tartaglia's name. Tartaglia refused, stating he would publish the solution himself at a later date. This prompted Cardano to label Tartaglia as greedy and unwilling to help mankind.

Because of these insults, a correspondence developed between the two mathematicians which resulted in Tartaglia visiting Cardano in Milan. During this visit Tartaglia relented and offered Cardano a cryptic poem containing a solution to $x^3+ax=b$, provided Cardano swore an oath that he would never reveal the solution. Cardano accepted the terms, but was unable to decipher the code, so he asked for and received the necessary clue from Tartaglia. Using the solution to $x^3+ax=b$, Cardano and Ferrari found solutions for $x^3+ax^2=b$, $x^3=ax^2+b$, and $x^3+b=ax^2$ by employing substitutions which reduced them to the known case.

In 1545, Cardano published the *Ars magna*, which contained Tartaglia's solution of the cubic with a statement that del Ferro and Tartaglia had each found solutions by independent research. Cardano also included some of his own discoveries, including the idea that every cubic should have three roots. Cardano also published Ferrari's solution to the biquadratic equation here, with due credit to Ferrari.

When he had seen the *Ars magna*, Tartaglia publicly denounced Cardano for breaking an oath sworn on the Gospels, and he ridiculed Cardano's mathematical ability.

Cardano disdained to refute the slur, but Ferrari attacked Tartaglia, charging that Tartaglia had built up his reputation by defaming others, had stolen one proof in his new book without giving credit, and in addition had at least one thousand errors in the text. Ferrari ended his published response by challenging Tartaglia to a public debate on mathematics and all related subjects.

Tartaglia answered with further insults and refused the debate on the grounds that Cardano knew the men who would be the judges. Perhaps he really feared a public debate because he stammered.

After a further exchange of insults, each proposed thirty-one questions which were exchanged, answered, and returned. However, no decision was reached, because each tore the other's answers to shreds.

Then for no given reason, Tartaglia accepted a debate to be held in Milan, Cardano's stronghold.[9]

On August 10th, 1548, Tartaglia and Ferrari met in combat, Cardano having left town.

[8] Due to the limited use of symbols and a lack of understanding of negative quantities, these equations had to be treated separately. For example, Cardano wrote $x^3+6x=3$ as cub^a p: 6 reb^z aeqlis 20. Other methods of writing equations during this period can be seen in D. E. Smith's *History of Mathematics*, II, pp. 427–431.

[9] Perhaps Brescia, Tartaglia's city, demanded it because of civic pride.

Very little is known about the actual debate, but it appears to have degenerated into an invective match, with Tartaglia doing most of the shouting. Tartaglia left after the first day, claiming to have won, although it seems Ferrari won by default.

An indication of Ferrari's triumph is that Tartaglia lost his teaching post in Brescia, and Ferrari was invited to lecture in Venice, Tartaglia's stronghold.

Tartaglia died in 1557 without publishing his solution to the cubic, and when an attempt was made to publish his unpublished papers, none could be found which even mentioned the solution to the cubic.

With Cardano's death in 1576, one of the most interesting and colorful episodes in the history of mathematics ended.

Letters to the editor

(Continued from page 148)

4. $15x^2-14x-8$
$15x^2+(-20x+6x)-8$
$5x(3x-4)+2(3x-4)$
$(5x+2)(3x-4)$

The factors of *abcd* (120) are

1×120
2×60
3×40
4×30
5×24
(6×20)

5. $6x^2+7x-5$
$6x^2+(+10x-3x)-5$
$2x(3x+5)-1(3x+5)$
$(2x-1)(3x+5)$

The factors of *abcd* (30) are

1×30
2×15
(3×10)

These examples illustrate very clearly how the trial-and-error method for factoring a quadratic polynomial is completely eliminated.

Respectfully submitted,
JOSEPH RICCIUTI
Vare Junior High School
Philadelphia, Pennsylvania

Dear Editor:

As Anna sang, "By your children you'll be taught." In each of my algebra classes at least one of the children came up with a response to a question that involved thinking through a situation that was heretofore unknown to me. I should like to pass on a sample for comment.

While solving an example in class, we arrived at $-y=63$. I then inquired, "How do we get the value of y itself?" This was the reply, "Since y is the additive inverse of $-y$, then the value of y is the additive inverse of 63, that is -63." For me and the class this seemed a most satisfying explanation.

The thought process did not evolve out of a vacuum. Early in the term I introduced the idea of additive inverse and used it in the development of the rules for operation with signed numbers. Here at least is one meaningful application of "modern mathematics" that naturally integrated into the traditional curriculum.

Yours very truly,
GLADYS B. RHEINS
Great Neck North Junior High School
Great Neck, New York

Dear Editor:

The October, 1960 issue of THE MATHEMATICS TEACHER on page 459 contained Philip Peak's review of an article by Akira Yoshida which was printed in our *Math Student*, January, 1960.

We were delighted to read this review. Akira Yoshida's article on "Fermat and the Theory of Numbers" is outstanding, and we are proud that it has been called to the attention of the readers of THE MATHEMATICS TEACHER.

However, a teacher reading the review would not be aware that Akira Yoshida is a *high school* student and that the *Math Student* is the student publication in mathematics at Brooklyn Technical H. S., Brooklyn, New York. We feel that this additional information will provide a complete review of the article, and we know that your readers will be interested in it.

Incidentally, our *Math Student* will celebrate its thirtieth anniversary in 1961.

Thank you,

ROBERT PERKUS
Acting Chairman, Mathematics Department
Brooklyn Technical High School
Brooklyn, New York

Reviews and evaluations

Edited by Kenneth B. Henderson, University of Illinois, Urbana, Illinois

Basic Concepts of Elementary Mathematics, William L. Schaaf (New York: John Wiley & Sons, Inc., 1960). Cloth, vii+386 pp., $5.50.

The success of the modernization program in secondary-school mathematics will before too long depend upon the foundations laid in elementary-school arithmetic. The elementary-school teacher must have an adequate understanding of the "why" as well as the "how" of the mathematics she is teaching. Her knowledge and understanding of the mathematics must go well beyond the scope of the elementary arithmetic she is teaching. This adequate background in the content of modern mathematics is absolutely essential if the elementary-school teacher is to give top quality instruction to pupils in the early grades.

Basic Concepts of Elementary Mathematics by William L. Schaaf is an excellent textbook for furnishing this adequate background in the content of mathematics to the elementary-school teacher. It is highly recommended as a subject-matter textbook for our teachers' colleges. It can also be used as a text for an in-service elementary-school teacher-training program in mathematics.

This book is written in an informal, clear manner, and it is distinctly modern in flavor. The mathematics in the text is sound. The logical concepts of sets, relations, and functions are carefully discussed and well illustrated. The presentation of the material maintains a fairly high standard of rigor throughout, but the material can be understood by the student with a minimum of readings.

The only disagreement I can find with the author is in the arrangement of the topic chapters. The first chapter deals with topics in modern mathematics, and this chapter is very well done. Sets, relations, systems, and models are discussed carefully and are well illustrated. The exercises are good and well graded. The next two chapters deal with elementary logic and geometry, and this is where my suggestion of change comes. I would prefer to save these chapters for a later part of the text. I would like to see Chapter 4, entitled "The Number Concept," as Chapter 2. Once again, the author does an accurate job in discussing such topics as: number versus numeral, counting numbers, operations with numbers, etc.

From this point the text proceeds very well with "Extensions of the Number System" as the next chapter topic. Here the author presents rational numbers, the irrationals, the real numbers, and the complex numbers. The definitions contained in this chapter are precise. The exercises are not too difficult and geared for teachers on the elementary-school level. They are well chosen and illustrate well the principles discussed. Here again I would suggest a change and introduce at this point the work in functions and graphs which is a later chapter in the text. This chapter is also extremely well done with the modern ideas of ordered pairs, domain, and range discussed in a lucid manner. The author does discuss exponential and logarithmic functions, and this could serve as a background for the next chapter which I would suggest as "Exponents and Logarithms." The work in this chapter could be strengthened with a previous discussion and background in exponential and logarithmic functions.

The chapter on mensuration, elementary logic, and geometry could now be grouped in this order after all the work in functions has been completed. The author shows how logic can tie in with geometric proofs. The inclusion of work in co-ordinate geometry also adds to the quality of this text. The two final chapters entitled "Interest and Present Value" and "Probability and Insurance" are also worthwhile topics for inclusion in a text of this kind.

I would like to suggest, in summarizing, that the above-mentioned changes be construed not as a criticism of this fine text but rather as a "teaching order" of topics in the use of this text in the training of elementary- and junior-high-school teachers. This book can give the elementary-school teacher an insight into the nature and structure of mathematics from the modern viewpoint.—*Jack Shapiro, Newton High School, Newtonville, Massachusetts.*

Classical Mathematics, A Concise History of the Classical Era in Mathematics, Joseph E. Hofmann (New York: Philosophical Library, 1959). Cloth, 159 pp., $4.75.

This book and its predecessor, *The History of Mathematics*, reviewed in this journal in March 1958, are abridged translations of Professor Hofmann's three-volume *Geschichte der Mathematik* (Berlin, 1953 and 1957). The German edition, reviewed by Professor Carl Boyer in *Isis* (1955 and 1958), is notable for important bibliographies, omitted in the English version, and for various indexes that in translation have become mere lists of names. This is serious, for Dr. Hofmann is an authority on the history of mathematics, particularly of the seventeenth and eighteenth centuries and especially on the work of Leibniz.

Like the first volume, *Classical Mathematics*

is not suited to the general reader. The student will do well to keep Dr. Boyer's *History of Analytic Geometry* and his *Concepts of the Calculus* close at hand. Professor Hofmann makes no concessions to the amateur.

An initial error was made in the choice of titles for the translations. The German text had the descriptive subtitles "From the Beginning to the Advent of Fermat and Descartes," "From Fermat and Descartes to the Discovery of the Calculus and the Development of the New Methods," and "From the Controversy Concerning the Calculus to the French Revolution." The first volume of the English version has the title *History of Mathematics*—a misleading generality—while the second and third volumes are combined under the title *Classical Mathematics*, a term that is not defined.

Like the first volume, *Classical Mathematics* is strong in its occasional summaries of generalizations. These are of great interest to anyone concerned with the antecedents of the mathematics of the nineteenth and twentieth centuries. The author gives considerable information regarding the relations between mathematicians and their difficulties of communication. The reader is conscious of the importance of publications of the learned societies when the pages begin to be peppered with references to PT (*Philosophical Transactions*), AE (*Acta Eruditorum*), and the like. These references are of great value.

The events in the Newton-Leibniz controversy are listed in detail and prompt us to compare this account with the English ones with which we are more familiar. In this, incidentally, Dr. Hofmann has used the Gregorian calendar throughout to keep the chronology clear, but to make these dates in harmony with those in the English accounts, each English date given here must be decreased by ten days.

It is unfortunate that the printer failed to follow the American practice of displaying equations and formulas on separate lines, set off from the text. The translation also offers a number of unintentional red herrings which indicate that the translator was not familiar with English mathematics usage. For example, we are told that Descartes' first independent discovery was the so-called Euler Theorem on polyhedra $e+f=k+2$. Here the translator has preserved the initial letters of the German names. The e represents Ecke, the vertices, and k stands for Kanten, the edges. A more serious instance is the use of the word "applicates" for "ordinates." Reference to Dr. Boyer's work shows that Descartes and Fermat used the Latin term *ordinatis applicata* which appeared in French in the seventeenth century as *appliquée* or as *ordinée*, with the latter term gaining general usage.

These comments are made with the regret that the original text has not fared better in translation and that its usefulness has been curtailed by what Dr. Boyer calls "bibliographic surgery."—*Vera Sanford, State University College of Education, Oneonta, New York.*

Fundamentals of College Algebra, William Durfee (New York: The Macmillan Company, 1960). Cloth, vii+250 pp., $4.50.

Fundamentals of College Algebra will meet the demands of many teachers. Many of the familiar laws of algebra are proved by the author or included in the exercises throughout the book. Proofs are based upon the postulates of an ordered field, definitions, and the equals relation. Discussions on the use of quantifiers, the meaning of equality, and abstract fields are included. Inequalities are developed logically and represented graphically in one and in two dimensions. The elementary properties of polynomials are presented. The author has treated most of the topics of a course in college algebra. Two notable omissions are determinants and probability.

The suitability of this book as a text will depend largely upon the degree of agreement between the teacher and the author on this statement from the preface, "I have tried to be as rigorous as possible for a book at this level. I believe, however, that rigor can be overdone to the point of deadening the student's interest in mathematics and obscuring for him the beauty and scope of the subject." While this reviewer is in general agreement with the statement, there are areas in which it seems that lack of rigor may lead to lack of understanding. Treatment of the associative laws is cited as such an area. These laws are stated on page 15. On page 16 we are told that it is customary to omit the parentheses, and that, "By extension, there is no ambiguity in the meaning of $a+b+c+d+e$ or similar expressions." Then, in showing that the distributive law can be extended to $a(b+c+d)$, the first step is to write $a([b+c]+d)$ with the reason, "parenthesis convention." With this convention and with so little practice with the associative laws, one might wonder why they were introduced at all.

In the preface, the author says that he has emphasized sets. This reviewer was disappointed to find that the operations on sets were not discussed, the set-builder notation was not used, and the definition of a function was in terms of "a correspondence that associates" rather than a set of ordered pairs. The first four pages are devoted to descriptions and problems of sets and subsets. The word "set" is used throughout the book.

The treatment of equations is good. On page 69, we learn that "An equation is a statement that asserts that two symbols are names for the same number." The possibility of a false assertion is presented and this leads to a discussion of quantifiers.

It is felt that some of the impact of the fine work with inequalities is lost because of a failure to shade the graphs.

Some readers will be distressed to find "the set of whole numbers is a subset of the set of integers" (page 5) and "$-a$ is called the negative of a." (page 27) These readers will be pleased to find that the word "simplify" has re-

placed "reduce" in the section on fractions.

It is difficult to determine when to be rigorous and when rigor is overdone. This author has tried, and his book is worth your consideration. —*Vern Hiebert, Oregon College of Education, Monmouth, Oregon.*

An Introduction to Modern Mathematics, Robert W. Sloan (Englewood Cliffs, N. J.: Prentice-Hall, Inc., 1960). Cloth, xi+73 pp., $3.75.

In the first fifty pages of this book is a brief discussion of many of the newer topics to be found in what I shall call an up-to-date course in elementary algebra. The last sixteen pages discuss topics that should be found in an up-to-date course in intermediate algebra. The word "Introduction" in the title is quite appropriate, and I believe the book will arouse enough interest to make the reader want to look further into the literature for more complete treatments of many of the topics.

The author states in the preface that the major purpose of the book is "the development of elementary mathematics from a modern point of view." For a *quick* look at some of the newer approaches to elementary mathematics, this book does very well, but it does not always present the whole story, and it was not the author's intention to give a detailed treatment of all the topics covered.

In Chapter 1 the distinction between a thing and its name is explained. The explanation is satisfactory, but the use of semiquotes in the text to indicate the symbol or name instead of the thing that the symbol names, is not always consistent with the explanation. The last sentence on page 3 reads:

> Smiles is the longest word in English because there is a 'mile' between the first and last letter.

For this riddle to make sense to me, the semiquotes should have been around the word smiles and not around the word mile. I have not used semiquotes in my sentence, but I believe it is clear what I am talking about. I agree that the distinction between things and their names must be made, but this requires careful writing. The distinction between number and numeral is important, and there are many places in elementary mathematics where it must be made clear to the student whether you are talking about the symbol or the thing that the symbol names.

Chapter 2 on sets and sentences is excellent, and the definitions are precise and simple. The problems can easily be used in an elementary algebra class. The discussions involving logic, truth tables, quantifiers, and axioms are easily understood.

Chapter 3 on numbers, equations, and graphs contains much material involving inequalities, absolute values, and equivalent equations that should be in an elementary algebra course. The set of problems on page 43 that involves finding simpler names for sets and the graphing of each set is excellent. On page 38 we see the expression, "trip of -1 units." This should be rephrased to say something about the "trip corresponding to -1" to be consistent with the language and ideas used earlier and also at the bottom of this same page.

In Chapter 4 on ordered pairs, functions, and relations, many ideas are covered in an incomplete way. The sections on ordered pairs of real numbers and their graphs, open sentences in two variables, and lines and linear equations contain good material for elementary algebra. The material on relations, functions, composition of functions, inverse functions, and principal values belongs in intermediate algebra. Teachers can certainly get an introduction to these ideas here but will need a fuller treatment before presentation to their students.

Chapter 5 on the bracket function, the exponential function, mathematical induction, and the inverse of the exponential function is just sketchy enough to make the reader want to find out more about these special functions.

My personal appraisal is that this book covers most of the topics being considered today by many groups seeking to develop a "modern" approach to algebra in high school. It certainly will indicate to any teacher, in the short space of 66 pages, many of the newer ideas that should be in high school algebra courses. The detailed treatment must be obtained elsewhere. In my opinion the weakest part is Chapter 1.

This book will profit any high school teacher who will read it and do all the problems, seeking out more detailed presentations to clarify points not fully developed here.—*W. Eugene Ferguson, Newton High School, Newtonville, Massachusetts.*

Notice of appointment

The Board of Directors of the National Council of Teachers of Mathematics has appointed Eunice Lewis to the position of vice president, secondary-school level, to fill the vacancy created by the recent resignation of William Glenn. Miss Lewis is associate professor of education at the University of Oklahoma, teaching mathematics classes in the University High School and supervising the mathematics teacher trainees.

• TIPS FOR BEGINNERS

Edited by Joseph N. Payne, University of Michigan, Ann Arbor, Michigan, and William C. Lowry, University of Virginia, Charlottesville, Virginia

Introducing the slide rule early

Earnest E. Fandreyer, Lake Forest Academy, Lake Forest, Illinois

The slide rule is usually introduced after the topic of logarithms. It seems reasonable that the student must know logarithms before he can understand a logarithmic scale such as the slide rule. This need not be the case, however. There are, in the writer's opinion, good arguments for an early introduction to the slide rule, and also good rebuttals to those who argue the slide rule can be taught properly only after a treatment of logarithms.

First, consider that fractions and square roots are taught before the rational and irrational number systems are discussed in full. Triangles, areas, volumes, and other geometric concepts are introduced before starting demonstrative geometry. This approach is often called the intuitive approach, and it ranks high among teaching methods as long as the pupil understands clearly what he is doing. Likewise, the slide rule may be introduced long before logarithms, provided there is a simple way to explain the underlying principle and provided the work does not consist solely of mechanical manipulation.

Second, the topic of logarithms is usually taught in second-year algebra, which in most high schools is an eleventh-grade course. If the slide rule is introduced after logarithms, there is little time left for the students to use the rule—for the majority, only five to seven months. It is for this reason that many schools do not teach the slide rule at all but prefer to save time for more work with logarithms.

Third, the slide rule is obviously most useful when many problems involving multiplication and division are encountered. This is the case in grades seven and eight. At some time during the seventh grade the student has become familiar with the four basic operations on whole numbers, decimals, and common fractions. Thereafter, during these two years, the usual topics include many applications of these operations, such as interest problems and simple area and volume problems. Multiplication and division are applied extensively to solve these problems. At this point, then, the slide rule can be introduced with advantage.

The question arises, "Is there a way to explain the slide rule to eighth-graders?" The principle of the slide rule is simple enough. Multiplication of numbers is accomplished by the addition of distances on the rule, division by the subtraction of distances. We do not violate this principle if we suppress the fact that the necessary instrument is a logarithmic scale.

The teacher first shows, or repeats, how to add and subtract numbers using two ordinary foot rulers or yardsticks. Quite naturally, the question arises, "Can we also multiply and divide by this simple device of adding and subtracting distances?" Eighth-graders realize quickly that this cannot be done with the ordinary rulers, but that a pair of "different rulers" is required. They respond willingly when challenged to "invent" the new scales for

multiplication and division. All they need for the work is some writing material and an ordinary ruler.

As with every scale, an origin and a unit must be chosen at random. The teacher should properly emphasize this fact. The unit is then produced until there is a sufficient number of intervals on the scale. This scale should be drawn twice, on two different strips of paper, each about 12 inches long. Naturally, the pupils want to mark the origin with "0." The first point, a unit's distance from "0," is marked "1," and the other distances "2," "3," "4," . . . as on an ordinary ruler. Reflection on the purpose of the new scale is therefore necessary. We want to multiply two numbers, a and b, by adding the corresponding distances on the scales. If we denote the distances by $d(a)$ and $d(b)$ respectively, then $a \cdot b$ corresponds to $d(a)+d(b)$. Thus $2 \cdot 1$ corresponds to $d(2)+d(1)$. But since $2 \cdot 1 = 2$, $d(2)+d(1)=d(2)$. So $d(1)$ must equal zero. It follows that "1" and not "0" must be written at the origin of the new scale. The pupils will thus understand that there is no need for a "0" on a multiplication scale.

The unit length is then the distance between "1" and a point labeled other than "1" or "0," which can be chosen at random. The pupils will usually choose "2" as the label for this second point. The second interval on the scale is obtained by producing the unit length a second time. So we have $d(2)+d(2)$ and this corresponds to $2 \cdot 2$ or 4. The end-point of the second interval must therefore be marked by "4"; and that of the third interval by "8," since we have $d(4)+d(2)$ corresponding to $4 \cdot 2$ or 8. This procedure is continued.

If a length of two inches is chosen for the distance from "1" to "2," then at ten inches we have "32." which is enough for our present purpose. For placing the "3," the pupils invariably take the mid-point between "2" and "4." But then $3 \cdot 3$ corresponds to $d(3)+d(3)$, and on our scale that yields "8."

After some hestitation, the pupils will agree that the "3" must be placed further to the right, nearer the "4." We must place it by guess. Adding the distances for 2, 4, 8, and 3, we then find points for 6, 12, 24, and 9, and then for 18 and 27. Subtracting the distance for 2 from that of 3 yields the point for 1.5.

The scale will work so far, unless we have placed the "3" extremely close to the "4." In that case the point for 27 may fall to the right side of 32 instead of to its left. As with the "3," the place for the "5" is chosen by guess slightly to the right of the mid-point between "4" and "6." We now have a way of checking our guess for the place of a number. Multiplying 5 by 3, we check to see if the place for 15 falls reasonably to the left of the place for 16. If not, we change the position of the "5." The "7" is placed in a manner similar to the placing of the "5." We then have places for all the integers from 1 to 10.

Even from such a crude scale the pupils will learn two things. First, it seems probable that a scale for multiplication and division can be made. Second, the scale is not linear; the numbers lie closer together at the right end and are spaced wider apart at the left end of the scale. The teacher may then conclude the introduction by showing a regular slide rule.

Brighter students will sometimes ask how the inaccuracy in finding the places for "3," "5," and "7" can be overcome. The following procedure supplies an answer. We marked off intervals of unit length from the origin, and obtained points for 2, 4, 8, 16, and 32. There were no numbers between 1 and 2 or between 2 and 4 whose points we could find using our unit of 2. This unit was obviously too big. Since we chose it at random, we may just as well take any other number for our units, for example 1.1, and make a new scale with this smaller unit, which for obvious reasons can be a smaller distance on the scale. Where we formerly obtained points for 2, 4, 8, . . . we now get points for 1.1, 1.12, 1.33 . . . .

If we make the distance between the origin, 1, and 1.1 half an inch, we have

at 3.5 inches 1.1^7 or 1.95,
at 6 inches 1.1^{12} or 3.14,
at 8.5 inches 1.1^{17} or 5.05, and
at 12 inches 1.1^{24} or 9.85.

This multiplication scale is already fairly accurate even though the points for the integers are only approximated.

A still smaller unit, such as 1.01 or 1.001, would enable us to make an extremely accurate scale. This usually answers the questions of eighth-graders and there is no need to mention logarithms. The whole introduction should not require more than one or two periods and allows the children to be active rather than to follow passively a prescription. Once they have made such scales for multiplication and division themselves, children also learn how to multiply and divide with them in the process.

The introduction of the slide rule as here explained is certainly not the only possible way. Yet from my experience, though confined to a few courses, I venture to say that it works out well with eighth-graders. We see that there is indeed a way to introduce the slide rule early. The procedure obscures nothing, but rather draws attention to important mathematical concepts, such as nonlinear scales, choice of a convenient unit, and approximation, all of which have numerous applications in everyday life.

Products of signed numbers: an inductive approach

by William G. Annett, Seaford High School, Seaford, New York

Too often our students complete an introduction to algebra with a feeling that mathematics is nothing more than a bag of tricks and manipulations that produce desired results and answers at the right time. To cultivate a wholesome and intelligent attitude toward mathematics, the student must be made aware of the patterns and order inherent in the subject. He must learn to recognize and classify patterns. Guided at first by the teacher, then on his own, he must draw conclusions from these patterns.

The topic of signed numbers lends itself readily to the observation of patterns. Presupposing a knowledge of the result obtained by multiplying two positive numbers, which can easily be demonstrated by successive additions and the number line, the average eighth- or ninth-grade student would have little difficulty in recognizing the patterns in the following multiplications.

$$\begin{array}{rrrrrrr} +6 & +6 & +6 & +6 & +6 & +6 & +6 \\ +6 & +5 & +4 & +3 & +2 & +1 & 0 \\ \hline \end{array}$$

The patterns are obvious. The multiplicands remain constant; the multipliers decrease by $+1$; the products decrease by $+6$.

Suppose we were to continue our series of multiplications to the right without interrupting our established patterns in multiplier and multiplicand, which in turn should not alter the pattern of the products. The multiplicands will remain 6. Since the multipliers still decrease by $+1$ they will become $-1, -2, -3, \cdots$. Following the pattern of our products, we will have $-6, -12, -18, \cdots$. Note that the products are obtained by continuing the established "product pattern" and not by actual multiplication.

$$\begin{array}{rrrrrr} +6 & +6 & +6 & +6 & +6 & +6 \\ -1 & -2 & -3 & -4 & -5 & -6 \\ \hline \end{array}$$

Since we may extend our series of multiplications as far to the left or right as we choose, we may decide that the product of a positive number and a negative number should be a negative number. If the commutative property is to hold, the sign of the product of a negative number multiplied by a positive number should also be negative.

$$(+2)(-3) = -6$$

$$(-3)(+2) = -6$$

The same approach can be used in the case of a negative multiplied by a negative, which is usually the most difficult for the student to understand.

-6	-6	-6	-6	-6	-6	-6
$+6$	$+5$	$+4$	$+3$	$+2$	$+1$	0
-6	-6	-6	-6	-6	-6	
-1	-2	-3	-4	-5	-6	

Once again, our multiplicand remains constant while the multipliers decrease by $+1$. The pattern established by the first row of products above is -36, -30, -24, $\cdots$. Continuing the established pattern in the second row, we obtain products of $+6$, $+12$, $+18$, $\cdots$. Again, we obtain products without actual multiplication. Thus we decide that the product of two negative numbers should be positive.

For the average student, these examples usually provide enough situations to accept the conclusions as valid. If any doubt remains, a more extensive list of multiplications might help.

First-degree equations in more than three variables

Robert L. Page, Nasson College, Springvale, Maine

The first method of solving simultaneous linear equations which the student encounters is usually that of elimination. While this method may not be the best one for solving systems involving a large number of equations, it is nevertheless valuable in that it is easily understood and furnishes a basis for the logical development of determinants.

It is unfortunate, then, that most algebra texts fail to warn the student of possible pitfalls in the use of the method of elimination. The typical text carefully shows the student how to eliminate one variable from two linear equations in two unknowns to obtain a single equation in one unknown, which the student can solve by methods already known to him.

The method of elimination is then extended to three equations in three unknowns by showing how to eliminate one unknown from one pair of equations and the same unknown from another pair of equations to obtain a system of two linear equations in two unknowns, which the student is able to solve.

At this point, most texts dismiss the solution of four or more linear equations in four or more unknowns with the remark that they can be solved in a similar manner by reducing them one step at a time to a system of two equations in two unknowns. There is usually no statement of caution concerning the manner in which the student selects the various pairs of equations from which to eliminate the unknowns. It is this error of omission which can, and often does, lead to trouble.

For example, a student confronted with the following system of four equations in four unknowns might choose to eliminate z from these pairs of equations: (1) and (2), (1) and (4), and (2) and (4).

$$(1) \qquad 2w+4x+3y+z=6$$

$$(2) \qquad -3w+7x+2y-z=0$$

$$(3) \qquad 7w+5x-4y-8z=-11$$

$$(4) \qquad 5w+3x+7y+z=-5$$

The student has certainly satisfied the implied instructions of eliminating one unknown from three pairs of equations, and he has obeyed the often explicit instructions to eliminate that unknown which demands the least amount of tedious multiplication. When he takes one more step and reduces this new system of three equations in three unknowns to a system of two equations, he is in for a rude surprise! The resulting two equations are identical.

At this point, the student may do one of two things, depending on his mathematical maturity. He may solve for one of the remaining unknowns in terms of the other and conclude that the system is a dependent one, or he may give up and go on to the next problem. To show the student where he has gone astray and why he has not encountered this trouble before, the use of the following notation may be helpful.

Let E_{uij} represent the equation obtained by eliminating the variable u from linear equations i and j. By definition, E_{uji} is equivalent to E_{uij}. E_{uvijk} represents an equation obtained by eliminating variables u and v from equations i, j, and k, etc. Thus, in our hypothetical example above, the selection of pairs of equations would produce E_{z12}, E_{z14}, and E_{z24}.

Now let another variable, say w, be eliminated from these three equations. No matter how the pairs of equations are selected, the result is E_{zw124}, that is, an equation obtained by eliminating z and w from the original equations 1, 2, and 4. Small wonder the student has arrived at identical equations by diverse routes! For, by neglecting equation 3 in the original system, he has attempted to solve a system of three equations in four unknowns, which cannot, in general, be uniquely solved.

To see why systems of two or three linear equations do not give rise to this difficulty, we have only to consider possible combinations of two and three equations taken two at a time. With two equations, we have no choice of pairs, since there is only one combination possible. With three equations taken two at a time, there are three combinations possible. The first pair must of course involve two of the three equations. The second pair selected must involve one of the equations of the first combination and the third equation, for otherwise it would be identical to the first pair. In other words, with two or three equations it is impossible to select pairs of equations which do not involve all the given equations.

With four or more equations, however, the number of possible pairs of equations greatly increases and the required number of pairs of equations may be selected without involving all the original equations. In a system of four equations, there are six possible combinations and three "incorrect" ways to choose the pairs. In a system of more than four equations, there is a correspondingly larger number of possible pairs of equations and even more chances for error.

Although a check of more than forty high school and college texts has failed to reveal a single warning to the student about this type of error, the solution is simply to caution the student to be careful to use each equation at least once in selecting pairs of linear equations from a system of any number of equations. Once he understands the necessity for this, he will have eliminated another needless source of error and confusion.

A column of unofficial comment

A note on "variable"

by H. Van Engen, University of Wisconsin, Madison, Wisconsin

The material that is being produced by the School Mathematics Study Group is known for its excellence even though each and every teacher may have his own reservations about various parts of the total program. However, there are certain positions taken by SMSG after considerable debate that some people who have been working constantly in the secondary field find hard to understand. Possibly the best explanation is that the books were written by human beings, and human beings have mind sets. Hence, this note is a note about mind sets and intuitions.

It is common knowledge that SMSG refused to use the place-holder definition of a variable as used by reputable logicians. Furthermore, SMSG ignored the experience of many teachers who have found that the place-holder definition makes a great deal of sense to the first-year algebra student. It is rather difficult to imagine why the logicians and the circle of experienced teachers should have been ignored. Ignoring experience and logic would not have been so bad if the definition SMSG has used would give the first-year algebra student a true picture of the use of a letter in his first introduction to the subject.

Let us repeat the definition of a variable given by SMSG before continuing with our comment. "In a given computation involving a variable, the variable is a numeral which represents a definite though unspecified number from a given set."[1]

[1] *Mathematics for High Schools, First Course in Algebra*, New Haven: Yale University, 1960, p. 37.

Consider the following.

The admissible set consists of the real numbers.

The open sentence is $x^2+1=0$.

If x represents a "definite though unspecified number from the admissible set," which number is it? Obviously no number from the admissible set "converts" the open sentence to a sentence that expresses a true proposition. Hence, according to the SMSG definition, x is not a variable. Then what is x in this instance? Is it the unknown quantity or the general number of the old algebra courses? Will the high school algebra students learn that the null set cannot be a solution set for a condition? We hope not.

Consider a second and even more elementary situation.

The admissible set consists of the positive integers.

The open sentence is $2x-1=4$.

Here again we find that x cannot be "a numeral which represents a definite though unspecified number from a given set." And, again, we have no name for x as used in this instance if we accept the SMSG definition of a variable. This situation is quite unsatisfactory.

There are other objections to the SMSG definition of a variable. If x is a numeral, then it should be possible to label open sentences as true or false. For example, $x-1=2$. Does this sentence represent a statement that is true or false? According to the SMSG definition it should be possible to label it as true or false because x is a numeral and a numeral is a symbol for a number. To say

that x is an unknown or unspecified number is resorting to unclear statements which really tell the first-year algebra student little or nothing.

Of course, the argument is that one cannot use such symbols as + between two other symbols unless the latter represent numbers. But this point is highly debatable. In the study of polynomials, the algebraist uses x as an indeterminate. In this case, coefficients become the important elements (taken from a domain D) and where " . . . nothing is assumed known about x (not even that it is an unknown element of D), x is usually called an indeterminate."[2] Hence the algebraist is very willing to use such expressions as $a+bx+cx^2$ and say that "nothing is assumed known about x." However, with nothing known about x, the plus signs still appear in the polynomial. From this point of view, the study of algebra becomes the study of forms. Perhaps this is one point of view that the high school student might take about the algebra he studies. If he does, he would have many first-class logicians and algebraists applauding him.

First-class mathematicians are first class because they have strong intuitions about their subject. But scholars who possess strong intuitions about mathematical entities do not always possess equally strong intuitions about the problems of learning mathematics. Certainly in producing good mathematics texts for the high school we need to make use of both kinds of intuitions.

[2] Birkhoff, Garrett, and MacLane, Saunders, *A Survey of Modern Algebra* (rev. ed.; New York: The Macmillan Company, 1953), p. 58.

Comments on a note on "variable"[1]

by E. G. Begle, Yale University, New Haven, Connecticut

Professor Van Engen's note is concerned with a topic which is important in any beginning course in algebra. It is therefore essential that certain misunderstandings in his note be corrected.

In the first place, it is not correct that the SMSG writers ignored logicians and the teachers who had successfully taught the place-holder definition of variable. In fact, in the preliminary outline prepared in the summer of 1958, the SMSG *First Course in Algebra* used the place-holder definition of variable. In the following summer, when the first version of the complete text was prepared, this matter was again discussed at great length before it was finally decided to adopt the definition quoted in part[2] by Professor Van Engen. Thus the advocates of the place-holder definition, far from being ignored, were considered very seriously by the SMSG writers.

In arriving at their decision, the SMSG writers agreed with Professor Van Engen that pedagogical experience and the advice of logicians should be taken into account. However, they felt it equally important to take into account what he does not mention: the use and practice of *mathematics*. There can be no doubt that all of us do in fact use variables as names of numbers much of the time, especially in

[1] Some of the SMSG writers were consulted in the preparation of this note, but the author accepts complete responsibility for all statements in it.

[2] Actually, no brief formal definition of variable is given in this text. The concept is first introduced (p. 34) by means of specific examples and then is discussed in a paragraph, one sentence of which is that quoted by Professor Van Engen.

the kinds of problems that are treated in a beginning algebra course. The attitude of the SMSG writers might be summarized by saying that you should not hide from the students the thoughts and the vocabulary of the mathematical world provided that (and this is an important proviso) the mathematical thoughts and vocabulary are both sound and teachable.

However, it must not be concluded that because the SMSG writers decided that they would like to try the approach to the concept of variable found in this SMSG text, it is therefore official SMSG policy that this is the *only* way to treat this concept. The following quotation from the first SMSG Newsletter outlines the position of SMSG: "We do not believe that any single curriculum is to be preferred to the exclusion of all others, and we intend the curriculum embodied in these (SMSG) textbooks to be no more than a sample of the kind of curriculum which we hope to see in our schools."

From classroom experience during the past academic year, when more than 5,000 students studied the preliminary version of the SMSG text *First Course in Algebra*, we can conclude that the SMSG treatment of variable is pedagogically feasible and satisfactory, but we see no reason for a universal mind set on this topic. The more good ways we have of teaching good mathematics, the better off we are.

In his discussion of the open sentence $x^2+1=0$, Professor Van Engen has evidently misunderstood the nature of an open sentence. According to the SMSG definition[3] of an open sentence, and according to all definitions known to this writer, the open sentence $x^2+1=0$ does *not* assert the existence in the given domain of a number that makes the sentence true. The question is precisely to *find* the truth set. For any real number x, the computation of x^2+1 is perfectly meaningful and the resulting sentence $x^2+1=0$ is also meaningful. However, it happens to be always false, so the truth set of the open sentence is the empty set.

From his discussion of this and the other examples, it appears that Professor Van Engen is trying to interpret the letter x in a sentence such as $x^2+1=0$ as a variable whose domain of definition is the truth set of the sentence. As he has so aptly observed, this point of view leads to serious trouble. On the other hand, even a casual reading of *First Course in Algebra* will reveal that the SMSG writers have not made this rather common mistake.

In his penultimate paragraph, Professor Van Engen calls attention to an extremely important point. In any beginning course in algebra the term variable is bound to be used in two distinct ways. No matter what the first definition of variable is, the student will later in the course encounter the use of variable as an indeterminate.[4] In the SMSG text this occurs when polynomials are first studied.

In fact, as the student goes on in mathematics he will meet the identity function and thus may learn another interpretation of variable in terms of a function. And if he continues further, he may meet logicians for whom place-holders are the best way to formalize the foundations of algebra.

We are in complete agreement with Professor Van Engen's final paragraph. It expresses another basic policy of SMSG, and a glance at the roster[5] of SMSG writers will show that the preparation of the SMSG texts was shared by first-class mathematicians and first-class teachers.

[3] Open sentences are first discussed on page 42 of *First Course in Algebra*.

[4] A discussion of these two uses of variables and the interrelationships between these uses can be found in *Structure of Elementary Algebra*, by Vincent Haag. See especially Chapter 6.

[5] *SMSG Newsletter* No. 4 contains a complete list of participants in the 1958 and 1959 writing sessions.

On "variable"—a rebuttal

by H. Van Engen, University of Wisconsin, Madison, Wisconsin

Before considering certain points in Professor Begle's reply to my "A Note on 'Variable,'" it is fitting to thank the editor of THE MATHEMATICS TEACHER for the opportunity to continue the discussion. In these days of changing curriculum, a forum is needed where the pros and cons of the many issues that confront mathematics education may be debated. Exchanges of opinion bring out the issues for all mathematics teachers.

The value of raising issues has been demonstrated by the work of UICSM. Focusing the attention of teachers on the problems confronting secondary mathematics is one of the many good things that UICSM has done for mathematics education. No doubt in its own way SMSG will focus teachers' attention on even more fundamental issues.

The second point to be made before coming to specific remarks about Professor Begle's "Comments on a Note on 'Variable'" concerns word usage. To have used the word "ignore" in my original note was unfortunate. It would have been better if the basic question had been phrased: "Why did SMSG decide not to use the place-holder definition of a variable in the light of the successful experience many teachers have had with this definition and in the light of the stand that logicians take concerning the definition of variable?" An unfortunate choice of words often muddies the waters. There was no intent to distract the attention of the readers of THE MATHEMATICS TEACHER from the basic issue or to cast reflections on SMSG's consideration of basic issues.

It is certainly true that the pattern of words used by mathematicians must be used, in so far as possible, in the instructional programs in elementary and secondary schools. For example, when proving theorems about integers in the junior high school, it is desirable, convenient, and necessary to use such terms as "If n is an even number, then $n+1$ is an odd number."[1] This must be a goal of mathematics instruction, and it is a goal of all serious attempts at curriculum revision that the writer knows about. Hence, the differences of opinion concerning the term "variable" cannot be centered entirely on mathematics and its objectives. The difference concerns method as well as mathematics.

It is essential in mathematical instruction that we say what we mean and that we say it without using fuzzy terminology. Furthermore, precise terminology is characteristic of good mathematics. Most certainly the criticisms about fuzzy terminology leveled at the older courses in mathematics are justified. From this point of view, let us look once more at the SMSG definition. It states that " . . . the variable is a numeral which represents a *definite though unspecified* number from a given set [italics added]." In the universe of integers, what is a definite though unspecified integer? Do the words *definite* and *unspecified* contradict one another? How could a teacher make clear to a junior high school pupil the meaning of a *definite though unspecified* number? Many people feel that this definition has the same difficulties that the definitions in the older texts in algebra suffered from, namely, fuzziness. Only those who know something about mathematics can decipher what the words might mean.

Since the differences under consideration involve methodology, the teacher of mathematics might well ask what the following sentence means and how are we to "get the point across" in first-year algebra.

[1] In phrases such as "n is an even number," is n really a number or is this an idiomatic expression?

It seems necessary to give a brief answer to this question to illustrate the power of the place-holder definition.

To keep this reply within the bounds of a note, it is necessary to skip some initial lessons and focus our attention on the key idea. We will speak in terms of numbers, variables, and conditions. We arrive at this terminology after having studied open sentences, place-holders, and numerals as replacements for place-holders.

In the discussion which follows, let the universe be the integers. We imagine that instruction has proceeded until the class is ready to consider conditions in two variables. Among the conditions studied, the class will find the following types:

a) $y>x$
b) $y=x-1$
c) $y+x=2x$
d) $y+x=x+y$

When the pupils graph the solution sets to the above conditions, they will find that some of the conditions, such as (d), have solution sets which are equal to the universe. The class also notices that statements such as the following are true about these conditions. For example, "For each replacement of a and b, $a+b=b+a$." After having become familiar with the idea, the class is told that mathematicians really don't write, "For each replacement of a and b, $a+b=b+a$." Instead, mathematicians use an elliptical expression and write,

"For each a and b, $a+b=b+a$."

This is only one of many ways that could be used to introduce the algebra class to statements about the properties of numbers. It is the writer's belief that it is psychologically sounder and mathematically better than the "unspecified though definite" approach used by SMSG.

For the sake of the readers of this journal, the writer must insist on repeating an argument used in his original note. It is essential that SMSG's definition be kept in mind to understand the thought in the argument. SMSG says that, " . . . the variable is a numeral which represents a definite though unspecified number from a given set." Consider an example that might occur in almost any algebra class.

The set: the integers
The condition: $x^2+1=0$
Question: is x the name of a *definite* though *unspecified integer?*
Answer: NO.
Remark: Since x does not meet the conditions in SMSG's definition, it is not a variable. What is x?

The uses of letters in mathematics to represent mathematical entities are probably too numerous to list. However, the teacher of elementary mathematics should not be misled by the statement that " . . . the term variable is bound to be used in two distinct ways." There must be at least three in the first year of high school algebra alone. Add to this the "indeterminate" of the more advanced algebra courses, and we have four uses.[2]

Are there more uses of letters? Consider the following:

a) Forms: a/b, x^2+1, and 2^x
b) Conditions: $y=2x+1$
c) Statements: for each a and b, $(a+b)^2=a^2+2ab+b^2$

Every first-year algebra student should be conscious of these three uses of the variable.

In the reform of the present-day mathematics courses in the schools, it is important to keep in mind that the way a mathematician thinks, the way he talks, and the things he considers important should be taught, in so far as possible, in the schools. However, we must remember that many things that are important to the learner of mathematics are not important to the mature mathematician. It is in the area of things that are important to the learner that SMSG has fallen short.

[2] See Neal A. McCoy, *Introduction to Modern Algebra* (Boston: Allyn and Bacon, Inc., 1960), p. 132, for a discussion of indeterminates and variables as they occur in algebra.

In closing, three important items are called to the readers attention:

1. In the reply to the original note, SMSG's definition of a variable was not defended on mathematical grounds. Why not?

2. The definition was defended on the basis that it had been tried out by many teachers and that it had been found satisfactory. On this basis, sixteenth-century mathematics for our schools can be defended because the demand for the reform of school mathematics did not come from the high schools.

3. Readers of THE MATHEMATICS TEACHER should not interpret this writer's position as being one in opposition to the SMSG program. There are many good things about the SMSG program. Most certainly the course from which the definition of variable under consideration has been taken must be considered as one of the best, if not *the* best, algebra program for the first year in high school. The writer would recommend that all high school teachers study the text and seriously consider using it during the coming year.

Your professional dates

The information below gives the name, date, and place of meeting with the name and address of the person to whom you may write for further information. For information about other meetings, see the previous issues of THE MATHEMATICS TEACHER. Announcements for this column should be sent at least ten weeks early to the Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington 6, D.C.

NCTM convention dates

THIRTY-NINTH ANNUAL MEETING

April 5–8, 1961
Conrad Hilton Hotel, Chicago, Illinois
Hobert Sistler, Morton High School West, 2400 Home Avenue, Berwyn, Illinois

JOINT MEETING WITH NEA

June 28, 1961
Atlantic City, New Jersey
M. H. Ahrendt, 1201 Sixteenth Street, N.W., Washington 6, D.C.

TWENTY-FIRST SUMMER MEETING

August 21–23, 1961
University of Toronto, Toronto, Canada
Father John C. Egsgard, C.S.B., St. Michael's College School, 1515 Bathurst Street, Toronto 10, Canada

Other professional dates

Mathematics Section, Maryland State Teachers Association

March 18, 1961
Stephens Hall, Towson State Teachers College, Towson, Maryland
W. Edwin Freeny, 507 Milford Mill Road, Baltimore 8, Maryland

Chicago Elementary Teachers Mathematics Club

March 20, 1961
Toffenetti's Restaurant, 65 West Monroe Street, Chicago, Illinois
Mildred C. Rogers, Warren Elementary School, 9210 South Chappel Avenue, Chicago 17. Illinois.

The Greater Cleveland Council of Teachers of Mathematics

March 21, 1961
South Euclid School, Cleveland, Ohio
Bessie Kisner, Strongsville High School, Strongsville, Ohio

Illinois Council of Teachers of Mathematics

March 25, 1961 Monticello College, Godfrey, Illinois
April 1, 1961 Southern Illinois University, Carbondale, Illinois
April 14, 1961 Eastern Illinois University, Charleston, Illinois
April 15, 1961 Western Illinois University, Macomb, Illinois
April 22, 1961 Illinois State Normal University, Normal, Illinois
April 29, 1961 Sterling Township High School, Sterling, Illinois
T. E. Rine, Illinois State Normal University, Normal, Illinois

The Ontario Association of Teachers of Mathematics and Physics

April 3–5, 1961
University of Toronto, Toronto, Canada
Donald N. Hazell, Don Mills C. I., 15 Donway E., Don Mills, Ontario, Canada

The Greater Cleveland Council of Teachers of Mathematics

April 18, 1961
Bedford High School, Bedford, Ohio
Bessie Kisner, Strongsville High School, Strongsville, Ohio

Men's Mathematics Club of Chicago and Metropolitan Area

April 21, 1961
YMCA Hotel, 826 South Wabash Avenue, Chicago, Illinois
Vernon R. Kent, 1510 South Sixth Avenue, Maywood, Illinois

The California Mathematics Council and *The California Science Teachers Association*

April 28–30, 1961 (registration deadline April 15, 1961)
Santa Monica, California
Miss Mary Lou Hood, 3414 North Eckart, South San Gabriel, California

Pennsylvania Council of Teachers of Mathematics

April 29, 1961
Clarion State College, Clarion, Pennsylvania
Earl F. Myers, 322 13th Avenue, New Brighton, Pennsylvania

Association of Mathematics Teachers of New York State

May 5–6, 1961
Hotel Syracuse, Syracuse, New York
Edward E. Sherley, Mt. Pleasant High School, Schenectady 3, New York

Michigan Council of Teachers of Mathematics

May 5–7, 1961
M. E. S. Camp, Saint Mary's Lake, Battle Creek, Michigan
Mrs. Marjorie Pickering, 5198 Coldspring, Birmingham, Michigan

Chicago Elementary Teachers Mathematics Club

May 8, 1961
Toffenetti's Restaurant, 65 West Monroe Street, Chicago, Illinois.
Mildred C. Rogers, Warren Elementary School, 9210 South Chappel Avenue, Chicago 17, Illinois

What's new?

BOOKS

High School

High School Mathematics: Unit 6, Geometry, University of Illinois Committee on School Mathematics. Urbana, Illinois: University of Illinois Press, 1960. Paper, student's edition, 453 pp., $2.00; teacher's edition, $4.00.

Understanding Arithmetic, Maytscherl W. Reckless. Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1961. Cloth ix+372 pp., $3.76.

College

Calculus, R. L. Jeffery. Toronto, Canada: University of Toronto Press, 1960. Cloth, xii+298 pp., $4.95.

Elementary Logic of Science and Mathematics, P. H. Nidditch. Glencoe, Illinois: The Free Press of Glencoe, 1960. Cloth, vii+371 pp., $4.00.

Introduction to Symbolic Logic, A. H. Basson and D. J. O'Connor. Glencoe, Illinois: The Free Press of Glencoe, 1960. Cloth, viii+175 pp., $3.00.

Introductory Formal Logic of Mathematics, P. H. Nidditch. Glencoe, Illinois: The Free Press of Glencoe, 1960. Cloth, vii+188 pp., $3.00.

Miscellaneous

Adventures in Algebra ("TutorText"), Norman A. Crowder and Grace C. Martin, Garden City, New York: Doubleday and Company, 1960. Cloth, 350 pp., $3.95.

The Arithmetic of Computers ("TutorText"), Norman A. Crowder. Garden City, New York: Doubleday and Company, 1960. Cloth, 472 pp., $3.95.

Classics in Science, A Course of Selected Reading by Authorities, E. N. Da C. Andrade. New York: Philosophical Library, 1960. Cloth, xxiii+322 pp., $6.00.

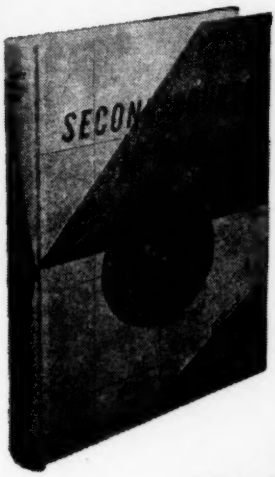
Compound Interest, N. E. Sheppard and D. C. Baille. Toronto: University of Toronto Press, 1960. Cloth, xix+225 pp., $5.95.

BOOKLETS

Miscellaneous

The Evolving Liberal Arts Curriculum: A Historical Review of Basic Themes, Willis Rudy. New York: Teachers College, Columbia University, 1960. Paper, 135 pp., $3.00.

Problem-Solving Methods in Science Teaching, Lester C. Mills and Peter M. Dean. New York: Bureau of Publications, Teachers College, Columbia University, 1960. Paper, viii+82 pp., $1.50.



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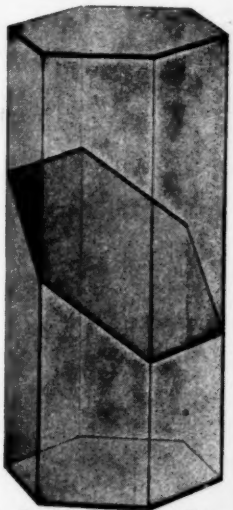

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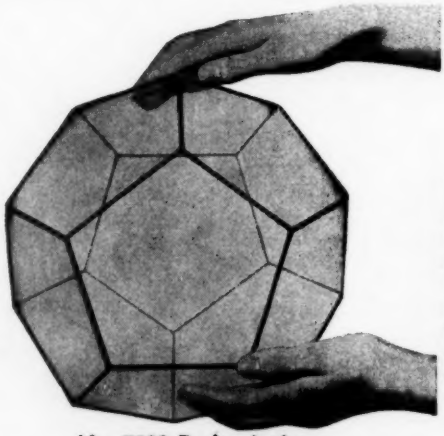

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